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Optimal additive quaternary codes of dimension 3.5 and 4

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Abstract

After the optimal parameters of additive quaternary codes of dimension $k \leq 3$ have been determined in [1], there was some activity to settle the next case of dimension $k = 3.5$ [2,3]. Here we complete dimensions $k = 3.5$ and $k = 4$. We also solve the problem of the optimal parameters of additive quaternary codes of arbitrary dimension when assuming a sufficiently large minimum distance.

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1 Introduction

A quaternary block code C of length n is a subset of $\mathbb{F}_4^n$. If C is closed under componentwise addition then C is called additive. If C is additive and closed under $\mathbb{F}_4$ scalar multiplication then C is called linear. The parameter k such that the number of codewords $|C|$ equals 4^k is called the dimension of C. Clearly, k is an integer if C is linear and a half-integer if C is additive. For each integer s let $n_k(s)$ denote the maximal length n such that an additive quaternary code of length n, dimension k, and minimum Hamming distance $n - s$ exists. For $k \leq 3$ the function $n_k(s)$ was completely determined in [1]. In the sequence of papers [2,3] the determination of $n_{3.5}(s)$ was narrowed down to $s \in \{6, 7, 12\}$, the example for $s = 13$ refers to [4]. Geometrically, $n_k(s)$ is the maximum number of lines in the projective space $\mathrm{PG}(2k-1, 2)$ such that each hyperplane contains at most s lines, see e.g. [1]. The aim of this paper is to completely determine $n_{3.5}(s)$, $n_4(s)$, and $n_k(s)$ for all sufficiently large s.

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The remaining part of the paper is structured as follows. In Section 2 we introduce the necessary preliminaries. The problem of the optimal parameters of additive quaternary codes of arbitrary dimension, assuming a sufficiently large minimum distance, is solved in Section 3, see Corollary 3.15. The determination of $n_{3.5}(s)$ and $n_4(s)$ is obtained in Section 4. It turns out that there exist infinite series of additive codes whose parameters outperform those of linear codes.

2 Preliminaries

The set of all subspaces of $\mathbb{F}_2^r$, ordered by the incidence relation $\subseteq$, is called $(r-1)$-*dimensional projective geometry over* $\mathbb{F}_2$ and denoted by $\mathrm{PG}(r-1,2)$. Employing this algebraic notion of dimension instead of the geometric one, we will use the term i-space to denote an i-dimensional subspace of $\mathbb{F}_2^r$. To highlight the important geometric interpretation of subspaces we will call 1-, 2-, and $(r-1)$-spaces points, lines, and hyperplanes, respectively. Every i-space in $\mathrm{PG}(r-1,2)$, where $r \geq i$, contains exactly 2^i-1 points. For two subspaces S and S' we write $S \subseteq S'$ if S is contained in S'. Moreover, we say that S and S' are *incident* iff $S \subseteq S'$ or $S \supseteq S'$.

Definition 2.1. An (n,r,s) *system* is a multiset $\mathcal{S}$ of n lines in $\mathrm{PG}(r-1,2)$ such that each hyperplane contains at most s elements from $\mathcal{S}$ and some hyperplane contains exactly s elements of $\mathcal{S}$. We say that $\mathcal{S}$ is *spanning* iff $s<n$.

By $n_k(s)$ we denote the maximum n such that a spanning $(n,2k,s)$ system exists, which is the same as the maximal length n of an additive quaternary code with dimension k and minimum Hamming distance $n-s$, see e.g. [1]. So, we will always assume $2(s+1) \geq k$ when considering $n_k(s)$.

Definition 2.2. For an (n,r,s) system $\mathcal{S}$ let $\mathcal{P}(\mathcal{S})$ denote the multiset of points that we obtain by replacing each element of $\mathcal{S}$ by its contained three points.

We also call a multiset of points spanning iff no hyperplane contains all points. If C is a binary linear code with length n and minimum Hamming distance d, then we say that C is an $[n,k,d]_2$ code, where $2^k=|C|$. Given a generator matrix G for C we can construct a multiset of points from C by considering the span of each column of G as a point in $\mathrm{PG}(k-1,2)$. And indeed, it is well known that a spanning multiset of points $\mathcal{P}$ in $\mathrm{PG}(k-1,2)$, such that at most s elements are contained in a hyperplane and some hyperplane contains exactly s elements, is in one-to-one correspondence to a linear $[n,k,n-s]_2$ code C, see e.g. [5, §1.1.2] or [6]. Let us write $\mathcal{P}=\mathcal{X}(C)$ and $C=\mathcal{X}^{-1}(\mathcal{P})$ for this correspondence. The

elements of a code are called *codewords*. The *weight* of a codeword $c \in C$ of a linear code is the number of non-zero entries in c. So, the minimum occurring non-zero weight of a linear code coincides with its minimum distance. We call a linear code Δ-*divisible* if the weights of all codewords are divisible by Δ. If Δ equals 2 or 4 then we also speak of even and doubly-even codes, respectively.

Lemma 2.3. *(Cf. [1, Lemma 1]) Let $\mathcal{S}$ be a spanning (n, r, s) system. Then, $C := \mathcal{X}^{-1}(\mathcal{P}(\mathcal{S}))$ is a 2-divisible $[3n, r, 2(n-s)]_2$ code with maximum weight at most $2n$.*

Proof. Since each line consists of $2^2 - 1 = 3$ points the cardinality of $\mathcal{P}(\mathcal{S})$ equals $3n$, so that C has length $3n$. Since $\mathcal{S}$ is spanning also $\mathcal{P}(\mathcal{S})$ is spanning and C has dimension r. Given an arbitrary hyperplane H in $\mathrm{PG}(r-1, 2)$ and an arbitrary line L we have that either L is completely contained in H or intersects the hyperplane in exactly a point. Since each hyperplane H contains $0 \leq i \leq s$ out of the n lines in $\mathcal{S}$ we have that H contains $3i + (n-i) = n + 2i$ points from $\mathcal{P}(\mathcal{S})$, so that the codeword $c \in C$ that corresponds to H has weight $(3n) - (n + 2i) = 2(n-i)$. □

So, bounds on the parameters of a binary linear code yield upper bounds for $n_k(s)$. E.g. the so-called *Griesmer bound* [7]

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{2^i} \right\rceil =: g(k, d) \tag{1}$$

relates the parameters of an $[n, k, d]_2$ code. If $n = g(k, d)$ we speak of Griesmer codes. Interestingly enough, Griesmer codes always exist if the minimum distance d is sufficiently large and a nice geometric construction was given by Solomon and Stiffler [8].

Lemma 2.4. *(Cf. [9, Theorem 12], [1, Lemma 1], and [2, Lemma 3]) Let $r > 2$ and $\mathcal{S}$ be an (n, r, s) system. Then, we have $g(r, 2(n-s)) \leq 3n$.*

Proof. Combine Lemma 2.3 with Inequality (1). □

In other words, we have

$$n \geq \left\lceil \frac{g(k, 2d)}{3} \right\rceil = \left\lceil \frac{\sum_{i=0}^{k-1} \left\lceil \frac{2d}{2^i} \right\rceil}{3} \right\rceil = \left\lceil \frac{2d + g(k-1, d)}{3} \right\rceil = d + \left\lceil \frac{g(k-1, d) - d}{3} \right\rceil, \tag{2}$$

where $d = n - s$ and $k = r$.

s	Griesmer upper bound	weak coding upper bound	$n_4(s)$
3	9	7	5
4	12		10
5	17		17
6	22	18	18
7	25	23	23
8	30	28	28
9	33		33
10	38	36	36
11	43	40	40
12	44		44
13	49		49
14	54		54
15	59	57	55
28	110		108
29	115		113

Table 1: The Griesmer and the weak coding upper bound for $n_4(s)$.

Definition 2.5. The *Griesmer upper bound* for $n_k(s)$ is the largest integer n such that $g(2k, 2(n-s)) \leq 3n$. The *weak coding upper bound* for $n_k(s)$ is the largest integer n such that a $[3n, 2k, 2(n-s)]_2$ code C exists. The *(strong) coding upper bound* for $n_k(s)$ is the largest integer n such that a 2-divisible $[3n, 2k, 2(n-s)]_2$ code C with maximum weight at most $2n$ exists.

We remark that the minimal possible length of an $[n, k, d]_2$ code is known for all $d \in \mathbb{N}$ when $k \leq 8$ [10], so that the weak coding upper bound can be easily evaluated for $n_4(s)$. This is different for the strong coding upper bound for $n_4(s)$, see Subsection 4.1.

Example 2.6. The Griesmer upper bound for $n_4(8)$ is 30 and the weak coding upper bound is 28. I.e., the Griesmer bound implies that no $[93, 8, 46]_2$ code exists but cannot rule out the existence of a $[90, 8, 44]_2$ code, so that $n_4(8) \leq 30$ is the sharpest upper bound we can deduce from the Griesmer bound (for linear codes). However, since the existence of a $[84, 8, 40]_2$ code and the non-existence of a $[87, 8, 42]_2$ code is known, we obtain $n_4(8) \leq 28$. In Table 1 we list the Griesmer and the weak coding upper bound for $n_4(s)$ for $3 \leq s \leq 15$ and all cases were either the weak coding upper bound is strictly less than the Griesmer upper

bound or $n_4(s)$ is strictly less than the weak coding upper bound. For $s \in \{3, 4\}$ we refer to [11]. Note that the cases $s \in \{1, 2\}$ cannot occur for a spanning $(n, 8, s)$ system. We do not display the weak coding upper bound when it coincides with the Griesmer upper bound.

In order to partially evaluate the strong coding upper bound we present a few tools from coding theory. Let C be a $[n, k, d]_2$ code with generator matrix G and $c \in C$ be a codeword of weight w. The *residual code of C with respect to c*, denoted by $\mathrm{Res}(C; c)$, is the code generated by the restriction of G to the columns where c has a zero entry. If only the weight w of c is relevant we will denote it by $\mathrm{Res}(C; w)$. The following statement on the residual code is well-known:

Lemma 2.7. *Let C be an $[n, k, d]_2$ code and let $d > \frac{w}{2}$. Then $\mathrm{Res}(C; w)$ is an $[n - w, k - 1, \geq d - \lfloor w/2 \rfloor]_2$ code.*

Lemma 2.8. *([12, Theorem 1]) Let C be an $[n, k, d]_2$ code with $n = g(k, d)$. If 2^e divides d, then C is 2^e-divisible.*

Proposition 2.9. *([13], MacWilliams identities) Let C be an $[n, k, d]_2$ code and $C^\perp$ be the dual code of C. Let $A_i(C)$ and $B_i(C)$ be the number of codewords of weight i in C and $C^\perp$, respectively. With this, we have*

$$\sum_{j=0}^{n} K_i(j) A_j(C) = 2^k B_i(C), \quad 0 \leq i \leq n \tag{3}$$

where

$$K_i(j) = \sum_{s=0}^{n} (-1)^s \binom{n-j}{i-s} \binom{j}{s}, \quad 0 \leq i \leq n$$

are the binary Krawtchouk polynomials. We will simplify the notation to A_i and B_i whenever C is clear from the context.

Whenever we speak of the first l MacWilliams identities, we mean Equation (3) for $0 \leq i \leq l - 1$. Adding the non-negativity constraints $A_i, B_i \geq 0$ we obtain a linear program where we can maximize or minimize certain quantities, which is called the linear programming method for linear codes. Adding additional equations or inequalities strengthens the formulation. For an $[n, k, d]_2$ code of full

length, i.e. $B_1 = 0$, we can rewrite the first four MacWilliams identities to

$$\sum_{i>0} A_i = 2^k - 1, \tag{4}$$

$$\sum_{i} iA_i = 2^{k-1}n, \tag{5}$$

$$\sum_{i} i^2A_i = 2^{k-1} \cdot (B_2 + n(n+1)/2)\,, \tag{6}$$

$$\sum_{i} i^3A_i = 2^{k-2} \cdot \big(3(B_2n - B_3) + n^2(n+3)/2\big)\,. \tag{7}$$

The weight enumerator can be generalized to the split weight enumerator based on a partition of the coordinates [14]. For a linear programming method based on the split enumerator we refer e.g. to [15].

Proposition 2.10. *([16, Proposition 5]) Let C be an $[n,k,d]_2$ code with all weights divisible by $\Delta := 2^a$ and let $(A_i)_{i=0,1,\ldots,n}$ be the weight distribution of C. Put*

$$\begin{aligned}\alpha &:= \min\{k-a-1, a+1\},\\ \beta &:= \lfloor (k-a+1)/2 \rfloor,\\ \delta &:= \min\{2\Delta i \mid A_{2\Delta i} \neq 0 \wedge i > 0\}.\end{aligned}$$

Then the integer

$$T := \sum_{i=0}^{\lfloor n/(2\Delta)\rfloor} A_{2\Delta i}$$

satisfies the following conditions:

1. *T is divisible by $2^{\lfloor (k-1)/(a+1)\rfloor}$.*
2. *If $T < 2^{k-a}$, then*
$$T = 2^{k-a} - 2^{k-a-t}$$
for some integer t satisfying $1 \le t \le \max\{\alpha, \beta\}$. Moreover, if $t > \beta$, then C has an $[n, k-a-2, \delta]_2$ subcode and if $t \le \beta$, it has an $[n, k-a-t, \delta]_2$ subcode.
3. *If $T > 2^k - 2^{k-a}$, then*
$$T = 2^k - 2^{k-a} + 2^{k-a-t}$$
for some integer t satisfying $0 \le t \le \max\{\alpha, \beta\}$. Moreover, if $a = 1$, then C has an $[n, k-t, \delta]_2$ subcode. If $a > 1$, then C has an $[n, k-1, \delta]_2$ subcode unless $t = a+1 \le k-a-1$, in which case it has an $[n, k-2, \delta]_2$ subcode.

For the constructive lower bound we have:

Lemma 2.11. *For $k > 1$ we have $n_k(s_1 + s_2) \geq n_k(s_1) + n_k(s_2)$ and $n_k(s+1) \geq n_k(s) + 1$.*

Proof. Let $\mathcal{S}_i$ be spanning $(n_i, 2k, s_i)$ systems for $i = 1, 2$ and $\mathcal{S}$ a spanning $(n, 2k, s)$ system. With this, the multiset union of $\mathcal{S}_1$ and $\mathcal{S}_2$ is a spanning $(n_1 + n_2, 2k, \leq s_1 + s_2)$ system. Adding an arbitrary line to $\mathcal{S}$ gives a spanning $(n + 1, 2k, \leq s + 1)$ system. □

Definition 2.12. A *vector space partition* of $\mathrm{PG}(r-1, 2)$ is a multiset $\mathcal{V}$ of subspaces with dimension at most $(r-1)$ such that every point of $\mathrm{PG}(r-1, 2)$ is contained in exactly one element of $\mathcal{V}$. We say that $\mathcal{V}$ has type $1^{t_1} 2^{t_2} \dots (r-1)^{t_{r-1}}$ if exactly t_i elements of $\mathcal{V}$ have dimension i for all $1 \leq i \leq r-1$.

A set of matrices $M \subseteq \mathbb{F}_2^{m \times n}$ with $\mathrm{rk}(A-B) \geq \delta$ for all $A, B \in M$ with $A \neq B$ is called a *rank metric code* with minimum rank distance δ. A Singleton-type upper bound gives $|M| \leq 2^{\max\{m,n\} \cdot (\min\{m,n\} - \delta + 1)}$. Rank metric codes attaining this bound are called *MRD* codes. They exist for all parameters with $\delta \leq \min\{m, n\}$, even if one additionally requires that M is linearly closed, see e.g. [17] for a survey.

Lemma 2.13. *For $r > 4$ there exists a vector space partition $\mathcal{V}$ of $\mathrm{PG}(r-1, 2)$ of type $2^{t_2}(r-2)^1$ where $t_2 = 2^{r-2}$.*

Proof. Let $M \subseteq \mathbb{F}_2^{2 \times (r-2)}$ be an MRD code with minimum rank distance 2 and cardinality 2^{r-2}. Prepending a 2×2 unit matrix to the elements of M gives generator matrices of 2-spaces in $\mathrm{PG}(r-1, 2)$ that are pairwise disjoint and disjoint to an $(r-2)$-space S. □

Lemma 2.14. *For $r > a > 2$ with $r \equiv a \pmod 2$ there exists a vector space partition $\mathcal{V}$ of $\mathrm{PG}(r-1, 2)$ of type $2^{t_2} a^1$ where $t_2 = 2^a \cdot \frac{2^{r-a}-1}{3}$.*

Proof. We prove by induction over r. Let $\mathcal{V}$ be the vector space partition obtained from Lemma 2.13 and let $S \in \mathcal{V}$ be the unique $(r-2)$-dimensional element. If $a = r-2$, which is indeed the case for all $r < 6$, then $\mathcal{V}$ is the desired vector space partition. Otherwise we identify S with $\mathrm{PG}(r-3, 2)$ and replace S by a vector space partition of $\mathrm{PG}(r-3, 2)$ of type $2^{t_2'} a^1$, which exists by induction. □

Lemma 2.15. *For $r > a > 2$ with $r \equiv a \pmod 2$ let $\mathcal{S}$ be the set of 2-dimensional elements of a vector space partition $\mathcal{V}$ of $\mathrm{PG}(r-1, 2)$ of type $2^{t_2} a^1$ and A be the unique a-dimensional element in $\mathcal{V}$. Then, $\mathcal{S}$ is a (t_2, r, s) system where $t_2 = 2^a \cdot \frac{2^{r-a}-1}{3}$ and $s = 2^{a-2} \cdot \frac{2^{r-a}-1}{3}$. Moreover, each hyperplane that contains A contains $s - 2^{a-2}$ elements from $\mathcal{S}$.*

Proof. Let H be an arbitrary hyperplane of $\mathrm{PG}(r-1,2)$. Note that every i-space intersects H in either 2^i-1 or $2^{i-1}-1$ points and that the elements of $\mathcal{S}$ partition the points outside of A. Counting points yields that H contains

$$\frac{2^{r-1}-2^{a-1}-t_2}{2} = 2^{a-2}\cdot\frac{2^{r-a}-1}{3} = s$$

elements from $\mathcal{S}$ if $A \subsetneq H$ and

$$\frac{2^{r-1}-2^{a}-t_2}{2} = \frac{2^{r-1}-2^{a-1}-2^{a-1}-t_2}{2} = s-2^{a-2}$$

elements from $\mathcal{S}$ if $A \subseteq H$. □

3 A generalization of the Solomon–Stiffler construction

In [8] Solomon and Stiffler constructed $[n,k,d]_2$ codes with $n=g(k,d)$ for all parameters with sufficiently large minimum distance d. Here we want to show the generalization that the Griesmer upper bound for $n_{k/2}(s)$ can always be attained if s is sufficiently large. Using a specific parameterization of the minimum distance d, the Griesmer bound in Inequality (1) can be written more explicitly:

Lemma 3.1. *Let k and d be positive integers. Write d as*

$$d = \sigma\cdot 2^{k-1} - \sum_{i=1}^{k-1}\varepsilon_i\cdot 2^{i-1}, \tag{8}$$

where $\sigma\in\mathbb{N}_0$ and $\varepsilon_i\in\{0,1\}$ for all $1\le i\le k-1$. Then, Inequality (1) is satisfied with equality iff

$$n = \sigma\cdot\left(2^k-1\right) - \sum_{i=1}^{k-1}\varepsilon_i\cdot\left(2^i-1\right), \tag{9}$$

which is equivalent to

$$n-d = \sigma\cdot\left(2^{k-1}-1\right) - \sum_{i=1}^{k-1}\varepsilon_i\cdot\left(2^{i-1}-1\right). \tag{10}$$

Given k and d, Equation (8) always determines σ and the ε_i uniquely. This is different for Equation (10) given k and $n-d=s$. Here it may happen that no solution with $0\le\varepsilon_i\le 1$ exists. By relaxing to $0\le\varepsilon_i\le 2$ we can ensure existence and uniqueness is enforced by additionally requiring $\varepsilon_j=0$ for all $j<i$ where $\varepsilon_i=2$ for some i. The same is true for Equation (9) given k and n. For more details we refer to [18, Chapter 2] which also gives pointers to Hamada's work on minihypers.

Definition 3.2. Let $\sigma \in \mathbb{N}$, $\varepsilon_1, \ldots, \varepsilon_{r-1} \in \mathbb{Z}$, and let V denote the r-dimensional ambient space $\mathrm{PG}(r-1,2)$. We say that an (n,r,s) system $\mathcal{S}$ has *type* $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ if there exist subspaces $S_1 \subseteq \cdots \subseteq S_{r-1}$ with $\dim(S_i) = i$ and

$$\sum_{S \in \mathcal{S}} \chi_S = \sigma \cdot \chi_V - \sum_{i=1}^{r-1} \varepsilon_i \cdot \chi_{S_i}, \tag{11}$$

where χ_S denotes the characteristic function of a subspace S, i.e., $\chi_S(P) = 1$ iff $P \subseteq S$ for every point P and $\chi_S(P) = 0$ otherwise. We say that $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ is *partitionable* if an (n,r,s) system with type $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ exists for suitable parameters n and s.

The notion of type $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ is motivated by the Solomon–Stiffler construction. E.g. $3[7] - 1[4] - 1[2]$ is an abbreviation for the construction taking all points of the ambient space $\mathrm{PG}(6,2)$ three times and subtracting the points of a line and a 4-dimensional subspace. The resulting multiset of points corresponds to a $[3 \cdot 127 - 1 \cdot 15 - 1 \cdot 3, 7, 3 \cdot 64 - 1 \cdot 8 - 1 \cdot 2]_2 = [363, 7, 182]_2$ code, which is optimal since it attains the Griesmer bound. Here we allow more flexibility by not restricting to $\varepsilon_i \in \{0,1\}$, so that the resulting codes might not be distance optimal. On the other hand we restrict the arrangement of the subspaces that are removed from (or added to) a suitable multiple of the ambient space for technical reasons. While simplifying the notation and allowing easier statements and proofs, some constructions are not covered by this definition.

Note that all chains $S_1 \subseteq \cdots \subseteq S_{r-1}$ are isomorphic, so that the notion of being partitionable does not depend on the choice of the subspaces $S_1, \ldots, S_{r-1}$. If $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ is partitionable, then also $0[r'] - (-\sigma)[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ is partitionable for all $r' > r$. The parameters of an (n,r,s) system can be computed from the parameters of a partition:

Lemma 3.3. *If $\mathcal{S}$ is an (n,r,s) system with type $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$, then we have*

$$n = \left(\sigma \cdot (2^r - 1) - \sum_{i=1}^{r-1} \varepsilon_i \cdot \left(2^i - 1\right) \right) / 3, \tag{12}$$

$$s = \max_{1 \le j \le r} \left(s_1 - \sum_{i=1}^{j-1} \varepsilon_i \cdot 2^{i-2} \right), \tag{13}$$

where

$$s_1 = \left(\sigma \cdot \left(2^{r-2} - 1\right) - \sum_{i=2}^{r-1} \varepsilon_i \cdot \left(2^{i-2} - 1\right) + \frac{1}{2} \cdot \varepsilon_1 \right) / 3. \tag{14}$$

Moreover, ε_1 is divisible by 2 and

$$\sum_{i=1}^{r-1} \varepsilon_i \cdot \left(2^i - 1\right) \equiv \sigma \cdot (2^r - 1) \pmod 3, \tag{15}$$

where the right hand side is congruent to zero modulo 3 if r is even.

Proof. Let $\mathcal{M}$ be the multiset of points covered by the elements of $\mathcal{S}$ and $S_1 \subseteq \cdots \subseteq S_{r-1}$ be subspaces as in Definition 3.2. Since $\mathcal{M}$ has cardinality

$$\sigma \cdot (2^r - 1) - \sum_{i=1}^{r-1} \varepsilon_i \cdot \left(2^i - 1\right)$$

and one line contains 3 points, we conclude Equation (12).

For an arbitrary hyperplane H let $1 \le j \le r$ denote the minimal integer such that $S_j \not\subseteq H$, where we set $j = r$ if $H = S_{r-1}$. Let $\mathcal{M}(H)$ denote the number of points of the multiset $\mathcal{M}$ restricted to hyperplane H. When extending the notion $\mathcal{M}(P)$ of the multiplicity of a point P in a multiset of points $\mathcal{M}$ additively to arbitrary subspaces S via $\mathcal{M}(S) := \sum_{P \le S} \mathcal{M}(P)$, this becomes a special case. Counting points gives

$$\begin{aligned}\mathcal{M}(H) &= \sigma \cdot \left(2^{r-1} - 1\right) - \sum_{i=1}^{j-1} \varepsilon_i \cdot \left(2^i - 1\right) - \sum_{i=j}^{r-1} \varepsilon_i \cdot \left(2^{i-1} - 1\right) \\ &= \sigma \cdot \left(2^{r-1} - 1\right) - \sum_{i=1}^{r-1} \varepsilon_i \cdot \left(2^{i-1} - 1\right) - \sum_{i=1}^{j-1} \varepsilon_i \cdot 2^{i-1}.\end{aligned}$$

The number s_j of elements of $\mathcal{S}$ contained in H is given by $\left(\mathcal{M}(H) - n\right)/2$, so that

$$\begin{aligned}s_j &= \Bigg(\sigma \left(2^{r-1} - 1\right) - \sum_{i=1}^{r-1} \varepsilon_i \left(2^{i-1} - 1\right) - \sum_{i=1}^{j-1} \varepsilon_i 2^{i-1} \\ &\qquad - \left(\sigma \left(2^r - 1\right) - \sum_{i=1}^{r-1} \varepsilon_i \left(2^i - 1\right)\right) \cdot \frac{1}{3}\Bigg)/2 \\ &= \left(\sigma \cdot \left(2^{r-1} - 2\right) - \sum_{i=1}^{r-1} \varepsilon_i \cdot \left(2^{i-1} - 2\right)\right)/6 - \sum_{i=1}^{j-1} \varepsilon_i \cdot 2^{i-2} \\ &= \left(\sigma \cdot \left(2^{r-2} - 1\right) - \sum_{i=2}^{r-1} \varepsilon_i \cdot \left(2^{i-2} - 1\right) + \frac{1}{2} \cdot \varepsilon_1\right)/3 - \sum_{i=1}^{j-1} \varepsilon_i \cdot 2^{i-2}.\end{aligned}$$

This verifies Equation (14) and yields

$$s_j = s_1 - \sum_{i=1}^{j-1} \varepsilon_i \cdot 2^{i-2} \tag{16}$$

for $2 \leq j \leq r$, which implies Equation (13). From $s_2 \in \mathbb{N}$ we conclude that ε_1 is divisible by 2. Equation (12) implies Equation (15) and $2^r - 1$ is divisible by 3 iff r is even. □

Corollary 3.4. *If all ε_i are nonnegative, then $s = s_1$ (using the notation from Lemma 3.3).*

Corollary 3.5. *If $\mathcal{S}_t$ is an (n_t, r, s_t) system with type $\left(\sigma + t \cdot \frac{3}{2^{\gcd(r,2)}-1}\right) \cdot [r] - \sum_{i=2}^{r-1} \varepsilon_i[i]$, where $\varepsilon_2, \ldots, \varepsilon_{r-1} \in \mathbb{N}$, then we have*

$$n_t = t \cdot \frac{2^r - 1}{2^{\gcd(r,2)} - 1} + \left(\sigma \cdot (2^r - 1) - \sum_{i=2}^{r-1} \varepsilon_i \cdot \left(2^i - 1\right)\right)/3, \tag{17}$$

$$s_t = t \cdot \frac{2^{r-2} - 1}{2^{\gcd(r,2)} - 1} + \left(\sigma \cdot \left(2^{r-2} - 1\right) - \sum_{i=2}^{r-1} \varepsilon_i \cdot \left(2^{i-2} - 1\right)\right)/3, \tag{18}$$

and

$$n_t - s_t = t \cdot \frac{3}{2^{\gcd(r,2)} - 1} \cdot 2^{r-2} + \sigma \cdot 2^{r-2} - \sum_{i=2}^{r-1} \varepsilon_i \cdot 2^{i-2}. \tag{19}$$

Next we state a few constructions.

Lemma 3.6. *If $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ and $\sigma'[r] - \sum_{i=1}^{r-1} \varepsilon'_i[i]$ are partitionable, then $(\sigma + \sigma') \cdot [r] - \sum_{i=1}^{r-1} (\varepsilon_i + \varepsilon'_i) \cdot [i]$ is partitionable.*

Proof. Fix some subspaces $S_1 \subseteq \cdots \subseteq S_{r-1}$ as in Definition 3.2. Let $\mathcal{S}$ be an (n, r, s) system with type $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ and $\mathcal{S}'$ be an (n', r, s') system with type $\sigma'[r] - \sum_{i=1}^{r-1} \varepsilon'_i[i]$, then the multiset union of the elements of $\mathcal{S}$ and $\mathcal{S}'$ is an (n'', r, s'') system with type $(\sigma + \sigma') \cdot [r] - \sum_{i=1}^{r-1} (\varepsilon_i + \varepsilon'_i) \cdot [i]$. □

A set of lines that partitions $\mathrm{PG}(r-1, 2)$ is called a *line spread*. They do exist iff r is even.

Lemma 3.7. *For $r > a \geq 2$ with $r \equiv a \pmod 2$ and $\sigma \in \mathbb{N}_{\geq 1}$ we have that $\sigma[r] - \sigma[a]$ is partitionable.*

Proof. If $a > 2$, then Lemma 2.15 yields the existence of an (n, r, s) system $\mathcal{S}$ with type $[r] - [a]$ and we can use σ copies thereof. For $a = 2$ we replace $\mathcal{S}$ by a line spread $\mathrm{PG}(r-1, 2)$ where we remove an arbitrary element. □

Theorem 3.8. *(Cf. [19, page 83], [20, Corollary 8], or [21, Lemma 2]) For each $r \geq 2$ we have that $[r]$ is partitionable if r is even and that $3[r]$ is partitionable if r is odd.*

Proof. The statement is obvious for $r = 2$ and for $r = 3$ we consider the set of all seven lines in $\mathrm{PG}(2, 2)$. From Lemma 3.7 we deduce that $[r] - [2]$ is partitionable for all even $r \geq 4$ and that $3[r] - 3[3]$ is partitionable for all odd $r \geq 5$, so that the statement follows from Lemma 3.6. □

Lemma 3.9. *If $x[r] - \sum_{i=2}^{r-1} \varepsilon_i[i]$ is partitionable for $x \in \{\sigma, \sigma'\}$ then*

$$\left(\sigma + t \cdot \frac{3}{2^{\gcd(r,2)} - 1}\right) \cdot [r] - \sum_{i=2}^{r-1} \varepsilon_i[i]$$

is partitionable for all $t \geq 0$ and we have $\sigma \equiv \sigma' \pmod{\frac{3}{2^{\gcd(r,2)}-1}}$.

Proof. Note that $2^{\gcd(r,2)} - 1$ equals 3 iff r is even and 1 iff r is odd, so that Theorem 3.8 and Lemma 3.6 imply the first statement. For even r the statement $\sigma \equiv \sigma' \pmod{\frac{3}{2^{\gcd(r,2)}-1}}$ is true since $\sigma, \sigma' \in \mathbb{N}$. For odd r we use Equation (15). □

Definition 3.10. We say that $\star[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ is partitionable if there exists an integer σ such that $\sigma[r] - \sum_{i=1}^{r-1} \varepsilon_i[i]$ is partitionable.

Theorem 3.11. *Let $r \geq 3$, $g := \gcd(r, 2)$, and $\varepsilon_2, \ldots, \varepsilon_{r-1} \in \mathbb{Z}$ such that*

$$\sum_{i=2}^{r-1} \varepsilon_i \cdot \left(2^i - 1\right) \equiv 0 \pmod{2^g - 1}. \tag{20}$$

Then $\star[r] - \sum_{i=2}^{r-1} \varepsilon_i[i]$ is partitionable.

Proof. Due to Theorem 3.8 and Lemma 3.6 it suffices to consider the case $\varepsilon_i \in \mathbb{N}$ for $2 \leq i \leq r-2$ while Equation (20) still holds. From Lemma 3.7 we conclude that $\varepsilon_i[r] - \varepsilon_i[i]$ is partitionable for all $i \equiv r \pmod 2$ and $2 \leq i < r$ as well as that $\varepsilon_i[r-1] - \varepsilon_i[i]$ is partitionable for all $i \equiv r-1 \pmod 2$ and $2 \leq i < r-1$. So using Lemma 3.6 we can assume $\varepsilon_i = 0$ for all $2 \leq i \leq r-2$ while Equation (20) still

holds. Reusing our first argument again we can additionally assume $\varepsilon_{r-1} \in \mathbb{N}$. If r is odd, then let $\mathcal{S}$ be a line spread of S_{r-1} (using the notation from Definition 3.2 and the construction from Theorem 3.8). Choosing each line in $\mathrm{PG}(r-1,2)$ ε_{r-1} times and removing an ε_i-fold copy of $\mathcal{S}$ shows that $\star[r] - \varepsilon_{r-1}[r-1]$ is partitionable. If r is even, then Equation (20) yields $\varepsilon_{r-1} \equiv 0 \pmod 3$. Now let $\mathcal{S}$ be a multiset of lines partitioning the 3-fold copy of the points of S_{r-1}. Choosing each line in $\mathrm{PG}(r-1,2)$ $\varepsilon_{r-1}/3$ times and removing an $\varepsilon_i/3$-fold copy of $\mathcal{S}$ shows that $\star[r] - \varepsilon_{r-1}[r-1]$ is partitionable. □

Definition 3.12. For integers $n > s \geq 1$ and $r > 2$ let the *surplus* be defined by

$$\theta(n,r,s) := 3n - g(r, 2(n-s)). \tag{21}$$

So the surplus is negative iff n is larger than the Griesmer upper bound for $n_{r/2}(s)$.

Lemma 3.13. *Let $n > s \geq 1$ and $r > 2$ be integers. If $\theta(n,r,s) \geq 0$, then there exists an*

$$\left(n + t \cdot \frac{2^r - 1}{2^{\gcd(r,2)} - 1}, r, s + t \cdot \frac{2^{r-2} - 1}{2^{\gcd(r,2)} - 1}\right)$$

system $\mathcal{S}_t$ for all sufficiently large t.

Proof. Setting $d' := 2(n-s)$ and $n' := g(r,d')$ we have $\theta(n,r,s) = 3n - n' \geq 0$. Due to Lemma 3.1 we can choose integers σ, $\varepsilon_1, \ldots, \varepsilon_{r-1}$, with $\sigma \geq 0$ and $0 \leq \varepsilon_i < 2$ for all $1 \leq i \leq r-1$, such that

$$d' = \sigma \cdot 2^{r-1} - \sum_{i=1}^{r-1} \varepsilon_i \cdot 2^{i-1} \tag{22}$$

and

$$n' = \sigma \cdot (2^r - 1) - \sum_{i=1}^{r-1} \varepsilon_i \cdot \left(2^i - 1\right). \tag{23}$$

Since d' is divisible by 2 we have $\varepsilon_1 = 0$. Let $\tau := \theta(n,r,s)$, $\sigma' := \sigma + \tau$, $\varepsilon'_{r-1} = \varepsilon_{r-1} + 2\tau$, and $\varepsilon'_i = \varepsilon_i$ for all $1 \leq i \leq r-2$, so that $\varepsilon'_i \in \mathbb{N}$ for all $1 \leq i \leq r-1$ and $\varepsilon'_1 = 0$. Note that

$$d' = \sigma' \cdot 2^{r-1} - \sum_{i=2}^{r-1} \varepsilon'_i \cdot 2^{i-1} \tag{24}$$

and

$$3n = \sigma' \cdot (2^r - 1) - \sum_{i=2}^{r-1} \varepsilon'_i \cdot \left(2^i - 1\right), \tag{25}$$

so that

$$\sum_{i=2}^{r-1} \varepsilon'_i \cdot \left(2^i - 1\right) \equiv 0 \pmod{2^{\gcd(r,2)} - 1}. \tag{26}$$

From Theorem 3.11 and Lemma 3.9 we conclude that $\left(\sigma' + t \cdot \frac{3}{2^{\gcd(r,2)}-1}\right) \cdot [r] - \sum_{i=2}^{r-1} \varepsilon'_i[i]$ is partitionable for all sufficiently large t. From Corollary 3.5, Equation (24), and Equation (25) we compute the stated parameters of $\mathcal{S}_t$. □

Theorem 3.14. *For all sufficiently large s we have that $n_{r/2}(s)$ attains the Griesmer upper bound, see Definition 2.5.*

Proof. Let $s_i := \frac{2^{r-2}-1}{2^{\gcd(r,2)}-1} - i$ for $0 \le i < \frac{2^{r-2}-1}{2^{\gcd(r,2)}-1}$ and n_i be the Griesmer upper bound for $n_{r/2}(s_i)$, i.e. $3n_i \ge g(r, 2(n_i - s_i))$ while $3\,(n_i + 1) < g(r, 2(n_i + 1 - s_i))$. Let $\sigma_i, \varepsilon_{1,i}, \ldots, \varepsilon_{r-1,i} \in \mathbb{N}$ with $\varepsilon_{j,i} < 2$ for all $1 \le j \le r-1$ be uniquely defined by

$$d_i := 2 \cdot (n_i - s_i) = \sigma_i \cdot 2^{r-1} - \sum_{j=1}^{r-1} \varepsilon_{j,i} \cdot 2^{j-1}, \tag{27}$$

so that

$$g(r, d_i) = \sigma_i \cdot (2^r - 1) - \sum_{j=1}^{r-1} \varepsilon_{j,i} \cdot \left(2^j - 1\right), \tag{28}$$

using Lemma 3.1, and $\theta\,(n_i, r, s_i) \ge 0$. Similarly, let $\sigma'_i, \varepsilon'_{1,i}, \ldots, \varepsilon'_{r-1,i} \in \mathbb{N}$ with $\varepsilon'_{j,i} < 2$ for all $1 \le j \le r-1$ be uniquely defined by

$$d'_i := 2 + d_i = \sigma'_i \cdot 2^{r-1} - \sum_{j=1}^{r-1} \varepsilon'_{j,i} \cdot 2^{j-1}, \tag{29}$$

so that

$$g(r, d'_i) = \sigma'_i \cdot (2^r - 1) - \sum_{j=1}^{r-1} \varepsilon'_{j,i} \cdot \left(2^j - 1\right) \tag{30}$$

and $\theta\,(n_i + 1, r, s_i) < 0$. Now, let $s_{i,t} := s_i + t \cdot \frac{2^{r-2}-1}{2^{\gcd(r,2)}-1}$ and $n_{i,t} := n_i + t \cdot \frac{2^r-1}{2^{\gcd(r,2)}-1}$, so that Lemma 3.1 implies

$$d_{i,t} := 2 \cdot (n_{i,t} - s_{i,t}) = \left(\sigma_i + t \cdot \frac{3}{2^{\gcd(r,2)} - 1}\right) \cdot 2^{r-1} - \sum_{i=1}^{r-i} \varepsilon_i \cdot 2^{i-1}, \tag{31}$$

$$g\left(r, d_{i,t}\right) = t \cdot \frac{2^r - 1}{2^{\gcd(r,2)} - 1} \cdot 3 + \sigma_i \cdot \left(2^r - 1\right) - \sum_{j=1}^{r-1} \varepsilon_{j,i} \cdot \left(2^j - 1\right), \tag{32}$$

$$d'_{i,t} := 2 + d_{i,t} = \left(\sigma'_i + t \cdot \frac{3}{2^{\gcd(r,2)} - 1}\right) \cdot 2^{r-1} - \sum_{i=1}^{r-i} \varepsilon'_i \cdot 2^{i-1}, \tag{33}$$

and

$$g\left(r, d'_{i,t}\right) = t \cdot \frac{2^r - 1}{2^{\gcd(r,2)} - 1} \cdot 3 + \sigma'_i \cdot \left(2^r - 1\right) - \sum_{j=1}^{r-1} \varepsilon'_{j,i} \cdot \left(2^j - 1\right). \tag{34}$$

Thus we have

$$\theta\left(n_{i,t}, r, s_{i,t}\right) = \theta\left(n_i, r, s_i\right) \geq 0 \text{ and } \theta\left(n_{i,t} + 1, r, s_{i,t}\right) = \theta\left(n_i + 1, r, s_i\right) < 0,$$

i.e. the Griesmer upper bound for $n_{r/2}\left(s_{i,t}\right)$ is given by $n_{i,t}$ for all $t \in \mathbb{N}$.

Lemma 3.13 yields the existence of an $\left(n_{i,t}, r, s_{i,t}\right)$ system $\mathcal{S}_{i,t}$ for all sufficiently large t. □

Corollary 3.15. *Assuming a sufficiently large $d \in \mathbb{N}$ and $2k \in \mathbb{N}_{\geq 3}$, an additive quaternary block code $C \subseteq \mathbb{F}_4^n$ with $|C| = 4^k$ and minimum Hamming distance d exists iff*

$$n \geq \left\lceil \frac{g(2k, 2d)}{3} \right\rceil = \left\lceil \frac{\sum_{i=0}^{2k-1} \left\lceil \frac{2d}{2^i} \right\rceil}{3} \right\rceil.$$

We remark that the statement of Theorem 3.14 was generalized to arbitrary additive codes over $\mathbb{F}_q$ in [22]. The proof of Theorem 3.14 suggests the following algorithm to determine explicit formulas for $n_{r/2}(s)$ assuming that s is sufficiently large. For all $0 \leq i < \frac{2^{r-2}-1}{2^{\gcd(r,2)}-1}$ compute the Griesmer upper bound n_i for $n_{r/2}(s_i)$ where $s_i = \frac{2^{r-2}-1}{2^{\gcd(r,2)}-1} - i$. Then we have

$$n_{r/2}\left(t \cdot \frac{2^{r-2} - 1}{2^{\gcd(r,2)} - 1} - i\right) = t \cdot \frac{2^r - 1}{2^{\gcd(r,2)} - 1} - \left(\frac{2^r - 1}{2^{\gcd(r,2)} - 1} - n_i\right) \tag{35}$$

for all sufficiently large t. As an example we mention:

Proposition 3.16. *(Cf. [2, Table I],[3, Table II]) For all sufficiently large t we have*

- $n_{3.5}(31t) = 127t$;
- $n_{3.5}(31t - 1) = 127t - 5$;

- $n_{3.5}(31t-2) = 127t-10$;
- $n_{3.5}(31t-3) = 127t-15$;
- $n_{3.5}(31t-4) = 127t-20$;
- $n_{3.5}(31t-5) = 127t-21$;
- $n_{3.5}(31t-6) = 127t-26$;
- $n_{3.5}(31t-7) = 127t-31$;
- $n_{3.5}(31t-8) = 127t-36$;
- $n_{3.5}(31t-9) = 127t-41$;
- $n_{3.5}(31t-10) = 127t-42$;
- $n_{3.5}(31t-11) = 127t-47$;
- $n_{3.5}(31t-12) = 127t-52$;
- $n_{3.5}(31t-13) = 127t-55$;
- $n_{3.5}(31t-14) = 127t-60$;
- $n_{3.5}(31t-15) = 127t-63$;
- $n_{3.5}(31t-16) = 127t-68$;
- $n_{3.5}(31t-17) = 127t-73$;
- $n_{3.5}(31t-18) = 127t-76$;
- $n_{3.5}(31t-19) = 127t-81$;
- $n_{3.5}(31t-20) = 127t-84$;
- $n_{3.5}(31t-21) = 127t-87$;
- $n_{3.5}(31t-22) = 127t-92$;
- $n_{3.5}(31t-23) = 127t-95$;
- $n_{3.5}(31t-24) = 127t-100$;
- $n_{3.5}(31t-25) = 127t-105$;
- $n_{3.5}(31t-26) = 127t-108$;
- $n_{3.5}(31t-27) = 127t-113$;
- $n_{3.5}(31t-28) = 127t-116$;
- $n_{3.5}(31t-29) = 127t-121$;
- $n_{3.5}(31t-30) = 127t-126$.

In [3] the stated formulas of Proposition 3.16 were indeed shown to be true for all $t \geq 2$ and $n_2(7,2;31-i)$ was determined for all $i \in \{0,\ldots,31\}\backslash\{19,24,25\}$, referring to [4] for $i = 18$ and [2] for the previous state of the art.

4 Exact values of $n_{3.5}(s)$ and $n_4(s)$

The existence of an (n,r,s) systems can be easily modeled as ILP (Integer Linear Programming) problem. Denoting the set of lines in $\mathrm{PG}(r-1,2)$ by $\mathcal{L}$ and

the set of hyperplanes in $\mathrm{PG}(r-1,2)$ by $\mathcal{H}$, an (n,r,s) system exists iff the ILP

$$\begin{aligned}\sum_{L\in\mathcal{L}} x_L &= n\\ \sum_{L\le H} x_L &\le s \quad \forall H\in\mathcal{H}\\ x_L &\in \mathbb{N} \quad \forall L\in\mathcal{L}\end{aligned}$$

admits a solution. To reduce the search space we prescribe subgroups of the automorphism group. Alternatively we can try to partition suitable multisets of points into lines. Those multisets of points can again be modeled as ILP problems and we may prescribe subgroups of the automorphism group, see e.g. [23]. Alternatively we use the database of *best known linear codes* (BKLC) in `Magma` or enumerate suitable linear codes using `LinCode` [24]. For the (known) conditions of the binary codes we refer to Lemma 2.3.

Theorem 4.1. *(Cf. [2, Table I],[3, Table II]) We have*

- $n_{3.5}(31t) = 127t$ *for* $t \ge 1$;
- $n_{3.5}(31t-1) = 127t-5$ *for* $t \ge 1$;
- $n_{3.5}(31t-2) = 127t-10$ *for* $t \ge 1$;
- $n_{3.5}(31t-3) = 127t-15$ *for* $t \ge 1$;
- $n_{3.5}(31t-4) = 127t-20$ *for* $t \ge 1$;
- $n_{3.5}(31t-5) = 127t-21$ *for* $t \ge 1$;
- $n_{3.5}(31t-6) = 127t-26$ *for* $t \ge 1$;
- $n_{3.5}(31t-7) = 127t-31$ *for* $t \ge 1$;
- $n_{3.5}(31t-8) = 127t-36$ *for* $t \ge 1$;
- $n_{3.5}(31t-9) = 127t-41$ *for* $t \ge 1$;
- $n_{3.5}(31t-10) = 127t-42$ *for* $t \ge 1$;
- $n_{3.5}(31t-11) = 127t-47$ *for* $t \ge 1$;
- $n_{3.5}(31t-12) = 127t-52$ *for* $t \ge 1$;
- $n_{3.5}(31t-13) = 127t-55$ *for* $t \ge 1$;
- $n_{3.5}(31t-14) = 127t-60$ *for* $t \ge 1$;
- $n_{3.5}(31t-15) = 127t-63$ *for* $t \ge 1$;
- $n_{3.5}(31t-16) = 127t-68$ *for* $t \ge 1$;
- $n_{3.5}(31t-17) = 127t-73$ *for* $t \ge 1$;
- $n_{3.5}(31t-18) = 127t-76$ *for* $t \ge 1$;
- $n_{3.5}(31t-19) = 127t-81$ *for* $t \ge 1$;
- $n_{3.5}(31t-20) = 127t-84$ *for* $t \ge 1$;
- $n_{3.5}(31t-21) = 127t-87$ *for* $t \ge 1$;

- $n_{3.5}(31t-22) = 127t-92$ *for* $t \geq 1$;
- $n_{3.5}(31t-23) = 127t-95$ *for* $t \geq 1$;
- $n_{3.5}(31t-24) = 127t-100$ *for* $t \geq 1$;
- $n_{3.5}(31t-25) = 127t-105$ *for* $t \geq 1$;
- $n_{3.5}(31t-26) = 127t-108$ *for* $t \geq 2$ *and* $n_{3.5}(5) = 17$;
- $n_{3.5}(31t-27) = 127t-113$ *for* $t \geq 2$ *and* $n_{3.5}(4) = 12$;
- $n_{3.5}(31t-28) = 127t-116$ *for* $t \geq 2$ *and* $n_{3.5}(3) = 7$;
- $n_{3.5}(31t-29) = 127t-121$ *for* $t \geq 2$;
- $n_{3.5}(31t-30) = 127t-126$ *for* $t \geq 2$.

Proof. We can assume $s \geq 3$. Theorem 3.8 yields $n_2(7,2;31t) = 127t$ for $t \geq 1$. In [11] $n_2(7,2;3) \leq 7$ was shown. The coding upper bound implies $n_2(7,2;4) \leq 12$ and $n_2(7,2;5) \leq 17$. All other upper bounds follow from the Griesmer upper bound. Due to Theorem 3.8 and Lemma 2.11 it suffices to give constructions for $s \in \{3,\ldots,13,15,21,25,26,30\}$. Using ILP searches we have found the following explicit constructions:[1]

$s=3$: $\binom{0100100}{0010101}$, $\binom{0101100}{0011111}$, $\binom{0101000}{0011010}$, $\binom{1000111}{0110101}$, $\binom{1001110}{0111111}$, $\binom{1001001}{0111010}$, $\binom{1100000}{0010000}$;

$s=4$: $\binom{1000111}{0010110}$, $\binom{1001110}{0011101}$, $\binom{1001001}{0011011}$, $\binom{1010110}{0111110}$, $\binom{1011101}{0111001}$, $\binom{1011011}{0110111}$, $\binom{1110100}{0000001}$, $\binom{1111100}{0000011}$, $\binom{1111000}{0000010}$, $\binom{0001010}{0000101}$, $\binom{0100000}{0010000}$, $\binom{1000000}{0100000}$;

$s=5$: $\binom{1000010}{0010101}$, $\binom{1000001}{0011101}$, $\binom{1001111}{0011001}$, $\binom{1000100}{0011010}$, $\binom{1001000}{0011011}$, $\binom{1000101}{0110111}$, $\binom{1001101}{0111100}$, $\binom{1001011}{0110011}$, $\binom{1001010}{0111110}$, $\binom{1001001}{0110110}$, $\binom{1010101}{0110010}$, $\binom{1011101}{0110001}$, $\binom{1011010}{0110100}$, $\binom{1011001}{0111111}$, $\binom{1011011}{0111000}$, $\binom{0100000}{0010000}$, $\binom{1000000}{0100000}$;

$s=6$: $\binom{0001001}{0000010}$, $\binom{0100011}{0000110}$, $\binom{0100010}{0010100}$, $\binom{0100110}{0011110}$, $\binom{0101101}{0010010}$, $\binom{0101100}{0010111}$, $\binom{1001100}{0000001}$, $\binom{1000011}{0011000}$, $\binom{1001011}{0011011}$, $\binom{1001110}{0010001}$, $\binom{1011010}{0000100}$, $\binom{1000110}{0100001}$, $\binom{1000001}{0110001}$, $\binom{1000010}{0110000}$, $\binom{1010111}{0100000}$, $\binom{1011101}{0110101}$, $\binom{1011000}{0111100}$, $\binom{1100010}{0001011}$, $\binom{1100101}{0001000}$, $\binom{1101010}{0011111}$, $\binom{1110001}{0001110}$, $\binom{1110100}{0001100}$;

$s=7$: $\binom{0010100}{0001010}$, $\binom{0011001}{0000101}$, $\binom{0010111}{0001111}$, $\binom{0100111}{0011010}$, $\binom{0101110}{0010101}$, $\binom{0101001}{0011111}$, $\binom{1000100}{0010001}$, $\binom{1001100}{0010011}$, $\binom{1001000}{0010010}$, $\binom{1000101}{0101011}$, $\binom{1001111}{0100110}$, $\binom{1001010}{0101101}$, $\binom{1010111}{0101000}$, $\binom{1011110}{0100100}$, $\binom{1010110}{0101010}$, $\binom{1011101}{0100101}$, $\binom{1011011}{0101111}$, $\binom{1011001}{0101100}$, $\binom{1010100}{0110001}$, $\binom{1011100}{0110011}$, $\binom{1011000}{0110010}$, $\binom{1100011}{0000111}$, $\binom{1100010}{0001110}$, $\binom{1100001}{0001001}$, $\binom{0001000}{0000100}$;

$s=8$: $\binom{0100001}{0010011}$, $\binom{0100011}{0010010}$, $\binom{0100010}{0010001}$, $\binom{0110101}{0001011}$, $\binom{0110111}{0001010}$, $\binom{0110110}{0001001}$, $\binom{1000001}{0011110}$, $\binom{1000011}{0011101}$, $\binom{1000010}{0011111}$, $\binom{1001101}{0010110}$, $\binom{1001111}{0010101}$, $\binom{1001110}{0010111}$, $\binom{1010101}{0101110}$, $\binom{1010111}{0101101}$, $\binom{1010110}{0101111}$, $\binom{1010001}{0111010}$, $\binom{1010011}{0111001}$, $\binom{1010010}{0111011}$, $\binom{1100101}{0011010}$, $\binom{1100111}{0011001}$, $\binom{1100110}{0011011}$, $\binom{1110001}{0000111}$, $\binom{1110011}{0000110}$, $\binom{1110010}{0000101}$, $\binom{0011000}{0000100}$, $\binom{0100000}{0010100}$, $\binom{1010100}{0001000}$, $\binom{1000000}{0110000}$, $\binom{1001100}{0111000}$, $\binom{1010000}{0101100}$, $\binom{1011000}{0111100}$, $\binom{1101000}{0010000}$;

$s=9$: $\binom{0100010}{0010111}$, $\binom{0100001}{0011100}$, $\binom{0101111}{0010110}$, $\binom{0100100}{0011110}$, $\binom{0101000}{0010011}$, $\binom{1000011}{0010010}$, $\binom{1001110}{0010001}$,

[1] We remark that constructions for $s=3,4$ were given in [11] and for $s=5,21$ we can use quaternary linear codes. For $s=9$ an example is given by a vector space partition of PG$(6,2)$ of type $2^{35}3^14^1$. For $s=15$ an example is given in [3, Example 2]. More constructions can e.g. be found in [3].

$\binom{1000110}{0010100}$, $\binom{1000111}{0011111}$, $\binom{1001100}{0011000}$, $\binom{1000010}{0100101}$, $\binom{1000001}{0101101}$, $\binom{1001111}{0101001}$, $\binom{1000100}{0101010}$, $\binom{1001000}{0101011}$,
$\binom{1000101}{0110010}$, $\binom{1001101}{0110001}$, $\binom{1001010}{0110100}$, $\binom{1001001}{0111111}$, $\binom{1001011}{0111000}$, $\binom{1010111}{0100011}$, $\binom{1011100}{0101110}$, $\binom{1010110}{0100111}$,
$\binom{1010011}{0101100}$, $\binom{1011110}{0100110}$, $\binom{1010101}{0111100}$, $\binom{1011101}{0110110}$, $\binom{1011001}{0110011}$, $\binom{1011011}{0111110}$, $\binom{1011010}{0110111}$, $\binom{1100001}{0011010}$,
$\binom{1101111}{0010101}$, $\binom{1100010}{0011011}$, $\binom{1100100}{0011001}$, $\binom{1101000}{0011101}$;

$s = 10$: $\binom{0100001}{0010110}$, $\binom{0101111}{0010011}$, $\binom{0100101}{0010001}$, $\binom{0101101}{0011111}$, $\binom{0100100}{0010111}$, $\binom{0100010}{0011100}$, $\binom{0101010}{0010010}$,
$\binom{0101011}{0010100}$, $\binom{0101000}{0011110}$, $\binom{0101001}{0011000}$, $\binom{1000111}{0010101}$, $\binom{1001100}{0011101}$, $\binom{1000011}{0011011}$, $\binom{1001110}{0011010}$, $\binom{1000110}{0011001}$,
$\binom{1010101}{0000010}$, $\binom{1011100}{0000001}$, $\binom{1010011}{0001000}$, $\binom{1011010}{0000100}$, $\binom{1010110}{0001111}$, $\binom{1000101}{0100111}$, $\binom{1001101}{0101100}$, $\binom{1001011}{0100011}$,
$\binom{1001010}{0101110}$, $\binom{1001001}{0100110}$, $\binom{1000010}{0110101}$, $\binom{1000001}{0111101}$, $\binom{1001111}{0111001}$, $\binom{1000100}{0111010}$, $\binom{1001000}{0111011}$, $\binom{1100101}{0000011}$,
$\binom{1100011}{0001110}$, $\binom{1101001}{0000111}$, $\binom{1100111}{0001100}$, $\binom{1101010}{0000110}$, $\binom{1110001}{0001010}$, $\binom{1111010}{0000101}$, $\binom{1110010}{0001011}$, $\binom{1110100}{0001001}$,
$\binom{1110101}{0001101}$;

$s = 11$: $\binom{0100010}{0010101}$, $\binom{0100001}{0011101}$, $\binom{0100101}{0010111}$, $\binom{0101101}{0011100}$, $\binom{0101111}{0011001}$, $\binom{0100100}{0011010}$, $\binom{0101011}{0010011}$,
$\binom{0101010}{0011110}$, $\binom{0101001}{0010110}$, $\binom{0101000}{0011011}$, $\binom{1000101}{0000010}$, $\binom{1001100}{0000001}$, $\binom{1000011}{0001000}$, $\binom{1001010}{0000100}$, $\binom{1000110}{0001111}$,
$\binom{1010010}{0000101}$, $\binom{1010001}{0001101}$, $\binom{1010011}{0001011}$, $\binom{1010100}{0001010}$, $\binom{1010110}{0001001}$, $\binom{1000001}{0111010}$, $\binom{1001111}{0110101}$, $\binom{1000010}{0111011}$,
$\binom{1000100}{0111001}$, $\binom{1001000}{0111101}$, $\binom{1010101}{0100111}$, $\binom{1011101}{0101100}$, $\binom{1011011}{0100011}$, $\binom{1011010}{0101110}$, $\binom{1011001}{0100110}$, $\binom{1100001}{0000110}$,
$\binom{1101100}{0000011}$, $\binom{1100011}{0000111}$, $\binom{1100010}{0001100}$, $\binom{1100110}{0001110}$, $\binom{1100101}{0010010}$, $\binom{1101101}{0010001}$, $\binom{1101010}{0010100}$, $\binom{1101001}{0011111}$,
$\binom{1101011}{0011000}$, $\binom{0100000}{0010000}$, $\binom{1000000}{0010000}$, $\binom{1000000}{0110000}$;

$s = 12$: $\binom{0010001}{0000110}$, $\binom{0010011}{0000101}$, $\binom{0010010}{0000111}$, $\binom{0100001}{0011110}$, $\binom{0100011}{0011101}$, $\binom{0100010}{0011111}$, $\binom{0101101}{0011011}$,
$\binom{0101111}{0011010}$, $\binom{0101110}{0011001}$, $\binom{1000001}{0001011}$, $\binom{1000011}{0001010}$, $\binom{1000010}{0001001}$, $\binom{1000101}{0110010}$, $\binom{1000111}{0110001}$, $\binom{1000110}{0110011}$,
$\binom{1001101}{0110011}$, $\binom{1001111}{0110010}$, $\binom{1001110}{0110001}$, $\binom{1010101}{0100101}$, $\binom{1010111}{0100110}$, $\binom{1010110}{0100101}$, $\binom{1010001}{0101010}$, $\binom{1010011}{0101001}$,
$\binom{1010010}{0101011}$, $\binom{1011101}{0110111}$, $\binom{1011111}{0110110}$, $\binom{1011110}{0110101}$, $\binom{1011001}{0111010}$, $\binom{1011011}{0111001}$, $\binom{1011010}{0111011}$, $\binom{1100101}{0011111}$,
$\binom{1100111}{0011110}$, $\binom{1100110}{0011101}$, $\binom{1101101}{0011011}$, $\binom{1101111}{0011010}$, $\binom{1101110}{0011001}$, $\binom{0010000}{0001000}$, $\binom{0100100}{0010100}$, $\binom{0110100}{0001000}$,
$\binom{1001100}{0010100}$, $\binom{1000100}{0100000}$, $\binom{1000000}{0111000}$, $\binom{1001000}{0111100}$, $\binom{1010100}{0100000}$, $\binom{1011100}{0100100}$, $\binom{1010000}{0111000}$;

$s = 13$: $\binom{0011010}{0000001}$, $\binom{0101110}{0000001}$, $\binom{0011000}{0000010}$, $\binom{1110001}{0000101}$, $\binom{0011001}{0000110}$, $\binom{1100011}{0001001}$, $\binom{1100110}{0001001}$,
$\binom{1000101}{0001010}$, $\binom{1010011}{0001010}$, $\binom{1100101}{0001101}$, $\binom{1110011}{0001101}$, $\binom{1010101}{0001110}$, $\binom{1110110}{0001111}$, $\binom{0101111}{0010000}$, $\binom{1000011}{0010001}$,
$\binom{1000111}{0010010}$, $\binom{1100111}{0010010}$, $\binom{1001000}{0010100}$, $\binom{1100010}{0010101}$, $\binom{1000001}{0010110}$, $\binom{1001000}{0010111}$, $\binom{0100111}{0011001}$, $\binom{0101000}{0011100}$,
$\binom{0101100}{0011100}$, $\binom{0101101}{0011101}$, $\binom{0100101}{0011110}$, $\binom{0101101}{0011110}$, $\binom{1000100}{0100000}$, $\binom{1010000}{0100001}$, $\binom{1001010}{0100011}$, $\binom{1010001}{0100011}$,
$\binom{1011000}{0100100}$, $\binom{1000000}{0100110}$, $\binom{1000110}{0101000}$, $\binom{1001101}{0101001}$, $\binom{1000110}{0101011}$, $\binom{1010000}{0101011}$, $\binom{1011101}{0101010}$, $\binom{1010011}{0110010}$,
$\binom{1000000}{0110101}$, $\binom{1001110}{0110111}$, $\binom{1010110}{0110110}$, $\binom{1000010}{0111000}$, $\binom{1000100}{0111001}$, $\binom{1010100}{0111000}$, $\binom{1001011}{0111011}$, $\binom{1011011}{0111010}$,
$\binom{1000011}{0111100}$, $\binom{1010111}{0111101}$, $\binom{1011110}{0111100}$, $\binom{1001100}{0111110}$;

$s = 15$: $\binom{0010101}{0001000}$, $\binom{0011001}{0000100}$, $\binom{0010101}{0001111}$, $\binom{0011001}{0000010}$, $\binom{0011010}{0000001}$, $\binom{0100011}{0000101}$, $\binom{0100011}{0001101}$,
$\binom{0100111}{0001001}$, $\binom{0100111}{0001011}$, $\binom{0100110}{0001010}$, $\binom{0100101}{0011000}$, $\binom{0101101}{0010100}$, $\binom{0101001}{0010010}$, $\binom{0101011}{0010001}$, $\binom{0101010}{0011111}$,
$\binom{1000010}{0010111}$, $\binom{1000001}{0011100}$, $\binom{1001111}{0010110}$, $\binom{1000100}{0011110}$, $\binom{1001000}{0010011}$, $\binom{1000110}{0100001}$, $\binom{1000011}{0101111}$, $\binom{1000111}{0100100}$,
$\binom{1001100}{0100010}$, $\binom{1000111}{0100101}$, $\binom{1001100}{0101101}$, $\binom{1000011}{0101011}$, $\binom{1001110}{0101010}$, $\binom{1001110}{0101000}$, $\binom{1000110}{0101001}$, $\binom{1000101}{0110110}$,
$\binom{1001101}{0110011}$, $\binom{1001011}{0110111}$, $\binom{1001010}{0111100}$, $\binom{1001001}{0111110}$, $\binom{1010100}{0100001}$, $\binom{1010010}{0101111}$, $\binom{1010001}{0101000}$, $\binom{1011111}{0100100}$,
$\binom{1011000}{0100010}$, $\binom{1010010}{0110101}$, $\binom{1010001}{0111101}$, $\binom{1010111}{0110010}$, $\binom{1011100}{0110001}$, $\binom{1010011}{0111000}$, $\binom{1011110}{0110100}$, $\binom{1011111}{0111001}$,
$\binom{1010100}{0111010}$, $\binom{1010110}{0111111}$, $\binom{1011000}{0111011}$, $\binom{1110001}{0000011}$, $\binom{1110001}{0001110}$, $\binom{1110010}{0000110}$, $\binom{1111000}{0000111}$, $\binom{1110100}{0001100}$,
$\binom{1000000}{0010000}$, $\binom{1000000}{0110000}$, $\binom{1010000}{0110000}$, $\binom{1100000}{0010000}$;

$s = 21$: $\binom{0010011}{0000100}$, $\binom{0011100}{0000010}$, $\binom{0010110}{0000001}$, $\binom{0010011}{0001111}$, $\binom{0010110}{0001000}$, $\binom{0010101}{0001100}$, $\binom{0011011}{0000110}$,

$\binom{0010101}{0001110}$, $\binom{0011010}{0000111}$, $\binom{0011001}{0000011}$, $\binom{0110001}{0000101}$, $\binom{0110010}{0001101}$, $\binom{0110011}{0000101}$, $\binom{0110011}{0001101}$, $\binom{0110001}{0001001}$,
$\binom{0110100}{0001011}$, $\binom{0110010}{0001010}$, $\binom{0110111}{0001001}$, $\binom{0110111}{0001011}$, $\binom{0110110}{0001010}$, $\binom{1000101}{0001000}$, $\binom{1001001}{0000100}$, $\binom{1000101}{0001111}$,
$\binom{1001001}{0000010}$, $\binom{1001010}{0000001}$, $\binom{1000011}{0010010}$, $\binom{1001110}{0010001}$, $\binom{1000110}{0010100}$, $\binom{1000111}{0011111}$, $\binom{1001100}{0011000}$, $\binom{1000010}{0100101}$,
$\binom{1000001}{0101101}$, $\binom{1000001}{0101011}$, $\binom{1001111}{0101010}$, $\binom{1000010}{0101001}$, $\binom{1001111}{0101001}$, $\binom{1000100}{0101010}$, $\binom{1000100}{0101101}$, $\binom{1001000}{0100101}$,
$\binom{1001000}{0101011}$, $\binom{1000111}{0110101}$, $\binom{1001100}{0111101}$, $\binom{1000011}{0111011}$, $\binom{1001110}{0111010}$, $\binom{1000110}{0111001}$, $\binom{1010101}{0100010}$, $\binom{1011101}{0100001}$,
$\binom{1010110}{0100011}$, $\binom{1010011}{0101110}$, $\binom{1010111}{0100010}$, $\binom{1011100}{0100001}$, $\binom{1010101}{0100111}$, $\binom{1011101}{0101100}$, $\binom{1010011}{0101000}$, $\binom{1011110}{0100100}$,
$\binom{1011110}{0100111}$, $\binom{1010111}{0101100}$, $\binom{1011100}{0100110}$, $\binom{1010110}{0101111}$, $\binom{1011011}{0100011}$, $\binom{1011010}{0101110}$, $\binom{1011001}{0100110}$, $\binom{1011010}{0100100}$,
$\binom{1011001}{0101111}$, $\binom{1011011}{0101000}$, $\binom{1010010}{0110101}$, $\binom{1010001}{0111101}$, $\binom{1011111}{0111001}$, $\binom{1010100}{0111010}$, $\binom{1011000}{0111011}$, $\binom{1100001}{0000011}$,
$\binom{1100001}{0001110}$, $\binom{1100010}{0000110}$, $\binom{1101000}{0000111}$, $\binom{1100100}{0001100}$, $\binom{1100101}{0010010}$, $\binom{1101101}{0010001}$, $\binom{1101010}{0010100}$, $\binom{1101001}{0011111}$,
$\binom{1101011}{0011000}$, $\binom{0100000}{0010000}$, $\binom{1000000}{0100000}$, $\binom{1010000}{0100000}$, $\binom{1010000}{0110000}$, $\binom{1100000}{0010000}$;

$s = 25$: $\binom{0010011}{0000101}$, $\binom{0010011}{0001101}$, $\binom{0010111}{0001001}$, $\binom{0010111}{0001011}$, $\binom{0010110}{0001010}$, $\binom{0010101}{0001100}$, $\binom{0011011}{0000110}$,
$\binom{0010101}{0001110}$, $\binom{0011010}{0000111}$, $\binom{0011001}{0000011}$, $\binom{0100001}{0010010}$, $\binom{0101111}{0010001}$, $\binom{0100010}{0010100}$, $\binom{0100011}{0011001}$, $\binom{0101110}{0011011}$,
$\binom{0100100}{0011000}$, $\binom{0100111}{0011010}$, $\binom{0101100}{0010101}$, $\binom{0100110}{0011101}$, $\binom{0101000}{0011111}$, $\binom{1000011}{0000100}$, $\binom{1001100}{0000010}$, $\binom{1000110}{0000001}$,
$\binom{1000011}{0001111}$, $\binom{1000110}{0001000}$, $\binom{1000010}{0010111}$, $\binom{1000001}{0011100}$, $\binom{1001111}{0010110}$, $\binom{1000100}{0011110}$, $\binom{1001000}{0010011}$, $\binom{1000011}{0100111}$,
$\binom{1001110}{0101100}$, $\binom{1000100}{0100101}$, $\binom{1000010}{0101101}$, $\binom{1000111}{0100110}$, $\binom{1001100}{0100011}$, $\binom{1000001}{0101001}$, $\binom{1001111}{0101011}$, $\binom{1000101}{0101010}$,
$\binom{1001101}{0100101}$, $\binom{1000110}{0101110}$, $\binom{1001001}{0101101}$, $\binom{1001000}{0101010}$, $\binom{1001011}{0101001}$, $\binom{1001010}{0101011}$, $\binom{1000010}{0110101}$, $\binom{1000001}{0111101}$,
$\binom{1000101}{0110010}$, $\binom{1001101}{0110001}$, $\binom{1000101}{0110111}$, $\binom{1001101}{0111100}$, $\binom{1001111}{0111001}$, $\binom{1000100}{0111010}$, $\binom{1001011}{0110011}$, $\binom{1001010}{0111110}$,
$\binom{1001001}{0110110}$, $\binom{1001010}{0110100}$, $\binom{1001001}{0111111}$, $\binom{1001000}{0111011}$, $\binom{1001011}{0111000}$, $\binom{1011011}{0100010}$, $\binom{1011110}{0100001}$, $\binom{1010110}{0100100}$,
$\binom{1010111}{0101111}$, $\binom{1011100}{0101000}$, $\binom{1010010}{0110101}$, $\binom{1010001}{0111101}$, $\binom{1010010}{0110111}$, $\binom{1010001}{0111100}$, $\binom{1010101}{0110010}$, $\binom{1011101}{0110001}$,
$\binom{1010110}{0110001}$, $\binom{1010011}{0111111}$, $\binom{1010111}{0110100}$, $\binom{1011100}{0110010}$, $\binom{1011111}{0110110}$, $\binom{1011111}{0111001}$, $\binom{1011110}{0111000}$, $\binom{1010100}{0111010}$,
$\binom{1010100}{0111110}$, $\binom{1011000}{0110011}$, $\binom{1011010}{0110100}$, $\binom{1011001}{0111111}$, $\binom{1011000}{0111011}$, $\binom{1011011}{0111000}$, $\binom{1100101}{0011000}$, $\binom{1101101}{0010100}$,
$\binom{1100101}{0011111}$, $\binom{1101101}{0011000}$, $\binom{1101001}{0010010}$, $\binom{1101011}{0010001}$, $\binom{1101010}{0010001}$, $\binom{1101010}{0011111}$, $\binom{1101011}{0010010}$, $\binom{1101001}{0010100}$,
$\binom{0100000}{0010000}$, $\binom{1000000}{0010000}$, $\binom{1000000}{0100000}$, $\binom{1000000}{0110000}$, $\binom{1010000}{0110000}$, $\binom{1100000}{0010000}$;

$s = 26$: $\binom{0010101}{0001000}$, $\binom{0011001}{0000100}$, $\binom{0010101}{0001111}$, $\binom{0011001}{0000010}$, $\binom{0011010}{0000001}$, $\binom{0100011}{0000101}$, $\binom{0100011}{0001101}$,
$\binom{0100111}{0001001}$, $\binom{0100111}{0001011}$, $\binom{0100110}{0001010}$, $\binom{0100101}{0001100}$, $\binom{0101011}{0000110}$, $\binom{0100101}{0001110}$, $\binom{0101010}{0000111}$, $\binom{0101001}{0000011}$,
$\binom{0100101}{0011000}$, $\binom{0101101}{0010100}$, $\binom{0101001}{0010010}$, $\binom{0101011}{0010001}$, $\binom{0101010}{0011111}$, $\binom{0110011}{0000101}$, $\binom{0110011}{0001101}$, $\binom{0110111}{0001001}$,
$\binom{0110111}{0001011}$, $\binom{0110110}{0001010}$, $\binom{1000011}{0000100}$, $\binom{1001100}{0000010}$, $\binom{1000110}{0000001}$, $\binom{1000011}{0001111}$, $\binom{1000110}{0001000}$, $\binom{1000101}{0100011}$,
$\binom{1001101}{0101100}$, $\binom{1001001}{0100111}$, $\binom{1001011}{0101100}$, $\binom{1001010}{0100110}$, $\binom{1000001}{0110010}$, $\binom{1001111}{0110001}$, $\binom{1000010}{0110001}$, $\binom{1000001}{0111111}$,
$\binom{1000010}{0110100}$, $\binom{1000100}{0110010}$, $\binom{1001111}{0111000}$, $\binom{1000100}{0111000}$, $\binom{1000101}{0111000}$, $\binom{1001101}{0110100}$, $\binom{1001001}{0110010}$, $\binom{1001011}{0110001}$,
$\binom{1001010}{0111111}$, $\binom{1001000}{0110100}$, $\binom{1001000}{0111111}$, $\binom{1010011}{0100100}$, $\binom{1011110}{0100010}$, $\binom{1010101}{0100001}$, $\binom{1011101}{0101111}$, $\binom{1010111}{0100001}$,
$\binom{1011100}{0101111}$, $\binom{1010101}{0100100}$, $\binom{1011101}{0100010}$, $\binom{1010110}{0101000}$, $\binom{1011001}{0100001}$, $\binom{1011011}{0101111}$, $\binom{1011010}{0100010}$, $\binom{1011011}{0100100}$,
$\binom{1011001}{0101000}$, $\binom{1011010}{0101000}$, $\binom{1010010}{0110101}$, $\binom{1010001}{0111101}$, $\binom{1010010}{0110111}$, $\binom{1010001}{0111100}$, $\binom{1010111}{0110101}$, $\binom{1011100}{0111101}$,
$\binom{1010011}{0111011}$, $\binom{1011110}{0111010}$, $\binom{1011111}{0110110}$, $\binom{1011111}{0111001}$, $\binom{1010100}{0111010}$, $\binom{1010110}{0111001}$, $\binom{1010100}{0111110}$, $\binom{1011000}{0110011}$,
$\binom{1011000}{0111011}$, $\binom{1100001}{0000011}$, $\binom{1100001}{0001110}$, $\binom{1100010}{0000110}$, $\binom{1101000}{0000111}$, $\binom{1100100}{0001100}$, $\binom{1100011}{0010010}$, $\binom{1101110}{0010001}$,
$\binom{1100110}{0010100}$, $\binom{1100101}{0011100}$, $\binom{1101101}{0010110}$, $\binom{1100101}{0011110}$, $\binom{1101101}{0010111}$, $\binom{1100111}{0011111}$, $\binom{1101100}{0011000}$, $\binom{1101001}{0010011}$,
$\binom{1101011}{0011110}$, $\binom{1101010}{0010011}$, $\binom{1101010}{0010111}$, $\binom{1101011}{0010110}$, $\binom{0101001}{0000010}$, $\binom{1001000}{0100000}$, $\binom{1000000}{0100000}$, $\binom{1000000}{0110000}$,
$\binom{1010000}{0100000}$, $\binom{1010000}{0110000}$, $\binom{1100000}{0010000}$;

$s = 30$: $\binom{0010001}{0000101}$, $\binom{0010010}{0001101}$, $\binom{0010011}{0000101}$, $\binom{0010011}{0001101}$, $\binom{0010001}{0001001}$, $\binom{0010100}{0001011}$, $\binom{0010010}{0001010}$,

$\binom{0010111}{0001001}$, $\binom{0010111}{0001011}$, $\binom{0010110}{0001010}$, $\binom{0100001}{0010101}$, $\binom{0101111}{0011101}$, $\binom{0100011}{0010100}$, $\binom{0101110}{0010010}$, $\binom{0100111}{0010001}$,
$\binom{0101100}{0011111}$, $\binom{0100010}{0011010}$, $\binom{0100100}{0011011}$, $\binom{0100110}{0011000}$, $\binom{0101000}{0011001}$, $\binom{0110101}{0000010}$, $\binom{0111100}{0000001}$, $\binom{0110011}{0001000}$,
$\binom{0111010}{0000100}$, $\binom{0110110}{0001111}$, $\binom{1000011}{0000100}$, $\binom{1001100}{0000010}$, $\binom{1000110}{0000001}$, $\binom{1000011}{0001111}$, $\binom{1000110}{0001000}$, $\binom{1000101}{0001100}$,
$\binom{1001011}{0000110}$, $\binom{1000101}{0001110}$, $\binom{1001010}{0000111}$, $\binom{1001001}{0000011}$, $\binom{1000101}{0011001}$, $\binom{1001101}{0011011}$, $\binom{1001011}{0010101}$, $\binom{1001010}{0011101}$,
$\binom{1001001}{0011010}$, $\binom{1000001}{0101010}$, $\binom{1001111}{0100101}$, $\binom{1000001}{0101011}$, $\binom{1001111}{0101010}$, $\binom{1000010}{0101001}$, $\binom{1000010}{0101011}$, $\binom{1000100}{0101001}$,
$\binom{1000100}{0101101}$, $\binom{1001000}{0100101}$, $\binom{1001000}{0101101}$, $\binom{1000011}{0110001}$, $\binom{1001110}{0111111}$, $\binom{1000010}{0110101}$, $\binom{1000001}{0111101}$, $\binom{1000110}{0110010}$,
$\binom{1001111}{0111001}$, $\binom{1000100}{0111010}$, $\binom{1000111}{0111000}$, $\binom{1001100}{0110100}$, $\binom{1001000}{0111011}$, $\binom{1010001}{0100010}$, $\binom{1011111}{0100001}$, $\binom{1010010}{0100100}$,
$\binom{1010101}{0100010}$, $\binom{1011101}{0100001}$, $\binom{1010110}{0100011}$, $\binom{1010011}{0101110}$, $\binom{1010101}{0100111}$, $\binom{1011101}{0101100}$, $\binom{1010111}{0100101}$, $\binom{1011100}{0101101}$,
$\binom{1010011}{0101011}$, $\binom{1011110}{0101010}$, $\binom{1011110}{0100111}$, $\binom{1010100}{0101000}$, $\binom{1010110}{0101001}$, $\binom{1010111}{0101100}$, $\binom{1011100}{0100110}$, $\binom{1011011}{0100011}$,
$\binom{1011010}{0101110}$, $\binom{1011001}{0100110}$, $\binom{1011010}{0100100}$, $\binom{1011001}{0101111}$, $\binom{1011000}{0101111}$, $\binom{1011011}{0101000}$, $\binom{1010010}{0110001}$, $\binom{1010001}{0111111}$,
$\binom{1010010}{0110101}$, $\binom{1010001}{0111101}$, $\binom{1010100}{0110010}$, $\binom{1011111}{0111001}$, $\binom{1011111}{0111000}$, $\binom{1010100}{0111010}$, $\binom{1010101}{0111100}$, $\binom{1011101}{0110110}$,
$\binom{1011001}{0110011}$, $\binom{1011011}{0111110}$, $\binom{1011000}{0110100}$, $\binom{1011010}{0110111}$, $\binom{1011000}{0111011}$, $\binom{1100001}{0000011}$, $\binom{1100001}{0001110}$, $\binom{1100010}{0000110}$,
$\binom{1101000}{0000111}$, $\binom{1100100}{0001100}$, $\binom{1100010}{0010111}$, $\binom{1100001}{0011100}$, $\binom{1100011}{0011010}$, $\binom{1101110}{0010101}$, $\binom{1101111}{0010110}$, $\binom{1100110}{0011011}$,
$\binom{1100100}{0011110}$, $\binom{1100111}{0011101}$, $\binom{1101100}{0011001}$, $\binom{1101000}{0010011}$, $\binom{0100000}{0010000}$, $\binom{1000000}{0010000}$, $\binom{1000000}{0100000}$, $\binom{1000000}{0110000}$,
$\binom{1010000}{0100000}$, $\binom{1010000}{0110000}$, $\binom{1100000}{0010000}$. □

Theorem 4.2. *For $s \geq 30$ the Griesmer upper bound for $n_4(s)$ can always be attained.*

- $n_4(21t) = 85t$ *for* $t \geq 1$;
- $n_4(21t-1) = 85t-5$ *for* $t \geq 1$;
- $n_4(21t-2) = 85t-10$ *for* $t \geq 1$;
- $n_4(21t-3) = 85t-15$ *for* $t \geq 1$;
- $n_4(21t-4) = 85t-20$ *for* $t \geq 1$;
- $n_4(21t-5) = 85t-21$ *for* $t \geq 1$;
- $n_4(21t-6) = 85t-26$ *for* $t \geq 2$ *and* $n_4(15) = 55$;
- $n_4(21t-7) = 85t-31$ *for* $t \geq 1$;
- $n_4(21t-8) = 85t-36$ *for* $t \geq 1$;
- $n_4(21t-9) = 85t-41$ *for* $t \geq 1$;
- $n_4(21t-10) = 85t-42$ *for* $t \geq 2$ *and* $n_4(11) = 40$;
- $n_4(21t-11) = 85t-47$ *for* $t \geq 2$ *and* $n_4(10) = 36$;
- $n_4(21t-12) = 85t-52$ *for* $t \geq 1$;
- $n_4(21t-13) = 85t-55$ *for* $t \geq 3$, $n_4(8) = 28$, *and* $n_4(29) = 113$;
- $n_4(21t-14) = 85t-60$ *for* $t \geq 3$, $n_4(7) = 23$, *and* $n_4(28) = 108$;
- $n_4(21t-15) = 85t-63$ *for* $t \geq 2$ *and* $n_4(6) = 18$;
- $n_4(21t-16) = 85t-68$ *for* $t \geq 1$;
- $n_4(21t-17) = 85t-73$ *for* $t \geq 2$ *and* $n_4(4) = 10$;
- $n_4(21t-18) = 85t-76$ *for* $t \geq 2$ *and* $n_4(3) = 5$;
- $n_4(21t-19) = 85t-81$ *for* $t \geq 2$;

- $n_4(21t-20) = 85t-84$ *for* $t \geq 2$.

Proof. We can assume $s \geq 3$. Theorem 3.8 yields $n_4(21t) = 85t$ for $t \geq 1$. In [11] $n_4(3) \leq 5$ and $n_4(4) \leq 10$ were shown. The weak coding upper bound implies $n_4(6) \leq 18$, $n_4(7) \leq 23$, $n_4(8) \leq 28$, $n_4(10) \leq 36$, and $n_4(11) \leq 40$. The strong coding upper bound implies $n_4(15) \leq 55$, $n_4(28) \leq 108$, and $n_4(29) \leq 113$, see Subsection 4.1. All other upper bounds follow from the Griesmer upper bound. For all

$$s \in \{5,\ldots,60\}\backslash\{9,10,11,14,15,23,24,27,28,29,44,45,49,50\}$$

the mentioned upper bound for $n_4(s)$ is matched by a quaternary linear code. We have $n_4(44) \geq n_4(21)+n_4(23) = 85+89 = 174$ and $n_4(45) \geq n_4(21)+n_4(24) = 85+94 = 179$. The following explicit examples were obtained using ILP searches:
$s = 9$: $\binom{01001011}{00010110}$, $\binom{01000110}{00111101}$, $\binom{01010101}{00000010}$, $\binom{01111110}{00000001}$, $\binom{01001101}{00101011}$, $\binom{01010010}{00100101}$, $\binom{01101110}{00011111}$, $\binom{01011001}{00111010}$, $\binom{01101001}{00000011}$, $\binom{10010000}{00000111}$, $\binom{10110000}{00001110}$, $\binom{10010001}{00101010}$, $\binom{10100110}{00010101}$, $\binom{10010101}{00101101}$, $\binom{10100100}{00011011}$, $\binom{10011100}{00110110}$, $\binom{10011101}{00111111}$, $\binom{10100000}{00001001}$, $\binom{10001000}{01011111}$, $\binom{10000100}{01111010}$, $\binom{10001100}{01100101}$, $\binom{10011000}{01100010}$, $\binom{10110100}{01010001}$, $\binom{10101100}{01110011}$, $\binom{11000110}{00011110}$, $\binom{11001101}{00111001}$, $\binom{11001011}{00100111}$, $\binom{11010000}{00100001}$, $\binom{11100011}{00010011}$, $\binom{11011100}{00100000}$, $\binom{11101000}{00010000}$, $\binom{11010010}{00110010}$, $\binom{11010100}{00110000}$;
$s = 10$: $\binom{00100101}{00001010}$, $\binom{00101100}{00011010}$, $\binom{00110010}{00001000}$, $\binom{01001100}{00100011}$, $\binom{01001010}{00110001}$, $\binom{01011000}{00100001}$, $\binom{01010001}{00111100}$, $\binom{01011100}{00111000}$, $\binom{01100010}{00011000}$, $\binom{01110001}{00000111}$, $\binom{10010000}{00000110}$, $\binom{10000000}{00111111}$, $\binom{10010100}{00100100}$, $\binom{10010101}{00101000}$, $\binom{10100000}{00000101}$, $\binom{10100100}{00001111}$, $\binom{10101001}{00000100}$, $\binom{10000010}{01001111}$, $\binom{10000011}{01011010}$, $\binom{10001011}{01011111}$, $\binom{10011101}{01001000}$, $\binom{10001001}{01110111}$, $\binom{10010111}{01100001}$, $\binom{10011110}{01101110}$, $\binom{10110010}{01010001}$, $\binom{10101110}{01111000}$, $\binom{10111110}{01100101}$, $\binom{10111100}{01101100}$, $\binom{10110011}{01110011}$, $\binom{11000010}{00001100}$, $\binom{11000011}{00101010}$, $\binom{11010111}{00110111}$, $\binom{11100010}{00011111}$, $\binom{11101000}{00010001}$, $\binom{11101010}{00010001}$, $\binom{11101011}{00011100}$;
$s = 11$: $\binom{00010011}{00001001}$, $\binom{11110101}{00001011}$, $\binom{00100110}{00001101}$, $\binom{00110110}{00001000}$, $\binom{11100011}{00000100}$, $\binom{01001001}{00000101}$, $\binom{01000100}{00011111}$, $\binom{11011010}{00100010}$, $\binom{01000011}{00110011}$, $\binom{11000000}{00101110}$, $\binom{01010100}{00110001}$, $\binom{11010010}{00111101}$, $\binom{01010111}{00110111}$, $\binom{11011100}{00110000}$, $\binom{01100011}{00001111}$, $\binom{01100000}{00010111}$, $\binom{11001100}{00110000}$, $\binom{01111100}{00000010}$, $\binom{11000110}{00000001}$, $\binom{10000111}{00001010}$, $\binom{10000100}{00010001}$, $\binom{10111101}{01000010}$, $\binom{10001000}{00100001}$, $\binom{10010100}{00100101}$, $\binom{10101111}{00011101}$, $\binom{10011101}{00101000}$, $\binom{10101101}{00010100}$, $\binom{10101011}{01011001}$, $\binom{10100001}{01011000}$, $\binom{10000001}{01010110}$, $\binom{10011110}{01001111}$, $\binom{10001011}{01011111}$, $\binom{10010010}{01001010}$, $\binom{10000110}{01100110}$, $\binom{10010110}{01110110}$, $\binom{10110011}{01110000}$, $\binom{10010001}{01111000}$, $\binom{10111111}{01111011}$, $\binom{10101000}{01100010}$, $\binom{10101110}{01101110}$;
$s = 14$: $\binom{01010000}{00100010}$, $\binom{01100001}{00010001}$, $\binom{01010011}{00100000}$, $\binom{01100010}{00010000}$, $\binom{01010001}{00110000}$, $\binom{01010011}{00110011}$, $\binom{10010110}{00001001}$, $\binom{11111000}{00000111}$, $\binom{10010101}{00100110}$, $\binom{11100010}{00011111}$, $\binom{10010001}{00111000}$, $\binom{11010111}{00100100}$, $\binom{10100100}{00001100}$, $\binom{11010100}{00001010}$, $\binom{10100010}{00011110}$, $\binom{11010001}{00111011}$, $\binom{10111011}{00000101}$, $\binom{11100110}{00001101}$, $\binom{10000001}{01011101}$, $\binom{10111010}{01111001}$, $\binom{10001010}{01010101}$, $\binom{10111000}{01111101}$, $\binom{10001001}{01011110}$, $\binom{10111100}{01111011}$, $\binom{10001110}{01011011}$, $\binom{10111101}{01110110}$, $\binom{10010011}{01000101}$, $\binom{10111111}{01001101}$, $\binom{10010100}{01000111}$, $\binom{10110010}{01001100}$, $\binom{10010000}{01001000}$, $\binom{10110100}{01000100}$, $\binom{10011011}{01001011}$, $\binom{10110000}{01000110}$, $\binom{10010111}{01011000}$, $\binom{10001000}{01110100}$, $\binom{10010101}{01011111}$, $\binom{10000101}{01111000}$, $\binom{10011110}{01010111}$, $\binom{10000111}{01111100}$, $\binom{10011101}{01011100}$, $\binom{10000011}{01111010}$, $\binom{10000010}{01100111}$, $\binom{10000110}{01101111}$, $\binom{10001100}{01100110}$, $\binom{10001101}{01101001}$, $\binom{10100111}{01100101}$, $\binom{10101001}{01100100}$, $\binom{10100110}{01101010}$, $\binom{10101010}{01101110}$, $\binom{10100001}{01001001}$, $\binom{10100000}{01001110}$, $\binom{10101011}{01001010}$, $\binom{10101111}{01001111}$;
$s = 23$: $\binom{00100001}{00010010}$, $\binom{00100000}{00010101}$, $\binom{00100101}{00010001}$, $\binom{00100110}{00010000}$, $\binom{00100011}{00011001}$, $\binom{00100010}{00011010}$, $\binom{00100100}{00011010}$, $\binom{00100111}{00011011}$, $\binom{01000001}{00010000}$, $\binom{01000111}{00010011}$, $\binom{01000100}{00010100}$, $\binom{01000101}{00010110}$, $\binom{01000011}{00011001}$, $\binom{01000000}{00011101}$, $\binom{01000000}{00011110}$, $\binom{01001001}{00011011}$, $\binom{01001000}{00011101}$, $\binom{01010110}{00001101}$, $\binom{01001010}{00111000}$, $\binom{01001100}{00111001}$, $\binom{01011001}{00110000}$, $\binom{01011000}{00110010}$, $\binom{01100110}{00011110}$,

$\binom{10000010}{00001110}$, $\binom{10000100}{00001001}$, $\binom{10000101}{00001010}$, $\binom{10000110}{00001100}$, $\binom{10000111}{00001111}$, $\binom{10000001}{00011100}$, $\binom{10001011}{00011100}$, $\binom{10010100}{00001000}$,
$\binom{10010000}{00110001}$, $\binom{10100100}{00011111}$, $\binom{10101101}{00010111}$, $\binom{10101001}{00011111}$, $\binom{10110111}{00001011}$, $\binom{10001001}{01000010}$, $\binom{10010011}{01000110}$, $\binom{10011000}{01001000}$,
$\binom{10000011}{01101011}$, $\binom{10000000}{01101101}$, $\binom{10001110}{01110001}$, $\binom{10011010}{01100000}$, $\binom{10011011}{01100101}$, $\binom{10011111}{01100100}$, $\binom{10010000}{01110110}$, $\binom{10010010}{01110111}$,
$\binom{10010110}{01110010}$, $\binom{10010001}{01111111}$, $\binom{10010101}{01111010}$, $\binom{10011000}{01110001}$, $\binom{10011001}{01110011}$, $\binom{10011110}{01111001}$, $\binom{10100010}{01001110}$, $\binom{10100110}{01001101}$,
$\binom{10101011}{01001011}$, $\binom{10101111}{01001111}$, $\binom{10110101}{01010111}$, $\binom{10111111}{01011100}$, $\binom{10111110}{01011111}$, $\binom{10100000}{01100011}$, $\binom{10100111}{01100001}$, $\binom{10100011}{01101111}$,
$\binom{10101000}{01101100}$, $\binom{10101110}{01101110}$, $\binom{10100101}{01110011}$, $\binom{10101010}{01111110}$, $\binom{10101100}{01111101}$, $\binom{10110001}{01100010}$, $\binom{10110010}{01100000}$, $\binom{10110000}{01100111}$,
$\binom{10110000}{01110101}$, $\binom{10110011}{01111011}$, $\binom{10110100}{01111100}$, $\binom{10111001}{01110100}$, $\binom{10111000}{01110110}$, $\binom{10111000}{01110111}$, $\binom{10111101}{01110100}$, $\binom{11000010}{00111011}$,
$\binom{11000001}{00111101}$, $\binom{11000111}{00111111}$, $\binom{11001010}{00110111}$, $\binom{11011011}{00101010}$, $\binom{11011001}{00101101}$, $\binom{11011010}{00101111}$, $\binom{11011100}{00101011}$, $\binom{11011111}{00101001}$,
$\binom{11011101}{00101110}$, $\binom{11011110}{00101100}$;

$s = 24$: $\binom{00010110}{00001101}$, $\binom{00100110}{00010111}$, $\binom{00100101}{00011111}$, $\binom{00101101}{00011101}$, $\binom{01000111}{00001010}$, $\binom{01000100}{00010100}$, $\binom{01010010}{00000101}$,
$\binom{01011011}{00000111}$, $\binom{01000110}{00101011}$, $\binom{01001110}{00101111}$, $\binom{01000000}{00110011}$, $\binom{01000100}{00111100}$, $\binom{01001001}{00111011}$, $\binom{01001100}{00111001}$, $\binom{01010001}{00100000}$,
$\binom{01010011}{00101000}$, $\binom{01010110}{00101100}$, $\binom{01011000}{00111000}$, $\binom{01011000}{00111111}$, $\binom{01100011}{00010101}$, $\binom{01100000}{00011100}$, $\binom{01101011}{00010010}$, $\binom{01101100}{00010000}$,
$\binom{01110111}{00001000}$, $\binom{10000111}{00001011}$, $\binom{10000011}{00010001}$, $\binom{10000000}{00011111}$, $\binom{10000010}{00011110}$, $\binom{10001110}{00010110}$, $\binom{10001111}{00011001}$, $\binom{10001011}{00100001}$,
$\binom{10001101}{00100010}$, $\binom{10000100}{00110010}$, $\binom{10001010}{00111101}$, $\binom{10010001}{00100010}$, $\binom{10010101}{00100011}$, $\binom{10011001}{00100111}$, $\binom{10010011}{00110100}$, $\binom{10011110}{00111000}$,
$\binom{10100100}{00010101}$, $\binom{10100101}{00011101}$, $\binom{10101001}{00011010}$, $\binom{10101011}{00011001}$, $\binom{10110111}{00001110}$, $\binom{10001001}{01000011}$, $\binom{10001000}{01001111}$, $\binom{10000101}{01010101}$,
$\binom{10000000}{01011111}$, $\binom{10011000}{01001010}$, $\binom{10011010}{01001000}$, $\binom{10011100}{01010100}$, $\binom{10011011}{01011001}$, $\binom{10011101}{01011101}$, $\binom{10000001}{01101001}$, $\binom{10000100}{01101000}$,
$\binom{10000110}{01101111}$, $\binom{10010000}{01101110}$, $\binom{10010111}{01111101}$, $\binom{10101101}{01001011}$, $\binom{10100000}{01011000}$, $\binom{10100010}{01011110}$, $\binom{10110001}{01000101}$, $\binom{10111011}{01000001}$,
$\binom{10111111}{01000010}$, $\binom{10110101}{01011010}$, $\binom{10100011}{01100101}$, $\binom{10100001}{01101010}$, $\binom{10110110}{01101111}$, $\binom{10110010}{01100010}$, $\binom{10110101}{01100110}$, $\binom{10110100}{01101010}$,
$\binom{10111010}{01101001}$, $\binom{10111101}{01101011}$, $\binom{10111100}{01101101}$, $\binom{11000011}{00001111}$, $\binom{11000001}{00011011}$, $\binom{11000011}{00011000}$, $\binom{11010001}{00001100}$, $\binom{11000010}{00100001}$,
$\binom{11000101}{00100000}$, $\binom{11001001}{00101000}$, $\binom{11001101}{00101010}$, $\binom{11001111}{00101001}$, $\binom{11001110}{00101110}$, $\binom{11010111}{00100011}$, $\binom{11011001}{00101100}$, $\binom{11010101}{00111110}$,
$\binom{11100010}{00000110}$, $\binom{11100111}{00001001}$, $\binom{11100100}{00010011}$, $\binom{11100101}{00011110}$, $\binom{11101000}{00010111}$, $\binom{11101101}{00010100}$, $\binom{11101100}{00011010}$;

$s = 27$: $\binom{00110101}{00000011}$, $\binom{01000011}{00011010}$, $\binom{01001010}{00010000}$, $\binom{01001000}{00011100}$, $\binom{01011011}{00000100}$, $\binom{01000001}{00101001}$, $\binom{01000110}{00110010}$,
$\binom{01001011}{00111101}$, $\binom{01001100}{00111111}$, $\binom{01001101}{00111110}$, $\binom{01010110}{00101111}$, $\binom{01011101}{00100000}$, $\binom{01010110}{00110100}$, $\binom{01100000}{00001111}$, $\binom{01100101}{00010010}$,
$\binom{01100111}{00011011}$, $\binom{01101011}{00010100}$, $\binom{01101101}{00011111}$, $\binom{01101110}{00011110}$, $\binom{10000100}{00011001}$, $\binom{10001000}{00101100}$, $\binom{10000001}{00111000}$, $\binom{10001101}{00110011}$,
$\binom{10001010}{00111100}$, $\binom{10010100}{00100100}$, $\binom{10010101}{00100100}$, $\binom{10010101}{00100110}$, $\binom{10010111}{00101000}$, $\binom{10011111}{00100011}$, $\binom{10010001}{00111111}$, $\binom{10010011}{00111110}$,
$\binom{10010100}{00111000}$, $\binom{10011000}{00110000}$, $\binom{10011010}{00110010}$, $\binom{10100000}{00000001}$, $\binom{10100110}{00011101}$, $\binom{10101010}{00011000}$, $\binom{10110101}{00000010}$, $\binom{10110011}{00001111}$,
$\binom{10110101}{00001000}$, $\binom{10111000}{00000101}$, $\binom{10001110}{01000010}$, $\binom{10001101}{01001110}$, $\binom{10000101}{01010001}$, $\binom{10000000}{01011111}$, $\binom{10001111}{01011110}$, $\binom{10010011}{01001111}$,
$\binom{10011011}{01000101}$, $\binom{10011100}{01000100}$, $\binom{10010010}{01011100}$, $\binom{10011000}{01010111}$, $\binom{10011111}{01011101}$, $\binom{10000111}{01101001}$, $\binom{10000110}{01110100}$, $\binom{10001001}{01110110}$,
$\binom{10001011}{01111101}$, $\binom{10010000}{01100001}$, $\binom{10010010}{01100110}$, $\binom{10011001}{01100011}$, $\binom{10010110}{01111011}$, $\binom{10011001}{01110000}$, $\binom{10100100}{01010101}$, $\binom{10100110}{01011011}$,
$\binom{10101000}{01010000}$, $\binom{10101011}{01011000}$, $\binom{10110010}{01000111}$, $\binom{10111000}{01000000}$, $\binom{10111001}{01001001}$, $\binom{10110001}{01010001}$, $\binom{10110110}{01010101}$, $\binom{10110000}{01011000}$,
$\binom{10111110}{01010010}$, $\binom{10101110}{01100100}$, $\binom{10100011}{01110001}$, $\binom{10100101}{01110101}$, $\binom{10100011}{01111110}$, $\binom{10100101}{01111010}$, $\binom{10101001}{01111010}$, $\binom{10111011}{01101100}$,
$\binom{10111111}{01101010}$, $\binom{10110111}{01111110}$, $\binom{11000100}{00010111}$, $\binom{11001000}{00010001}$, $\binom{11001101}{00010110}$, $\binom{11000000}{00100010}$, $\binom{11000110}{00101101}$, $\binom{11000111}{00110011}$,
$\binom{11000001}{00111010}$, $\binom{11000101}{00111011}$, $\binom{11001011}{00110100}$, $\binom{11010111}{00100001}$, $\binom{11011101}{00100111}$, $\binom{11011001}{00101110}$, $\binom{11010001}{00110101}$, $\binom{11010100}{00110001}$,
$\binom{11010110}{00110111}$, $\binom{11100110}{00001001}$, $\binom{11100111}{00010101}$, $\binom{11101010}{00010011}$, $\binom{10000011}{00111001}$, $\binom{10011110}{00101010}$, $\binom{10101111}{01010011}$, $\binom{10100010}{01111000}$,
$\binom{10000011}{00111001}$, $\binom{10011110}{00101010}$, $\binom{10101111}{01010011}$, $\binom{10100010}{01111000}$;

$s = 49$: $\binom{00010001}{00001010}$, $\binom{00010110}{00001001}$, $\binom{00010010}{00001100}$, $\binom{00010011}{00001110}$, $\binom{00010111}{00001011}$, $\binom{00010101}{00001111}$, $\binom{00010100}{00001101}$,
$\binom{00100010}{00001011}$, $\binom{00100001}{00001111}$, $\binom{00100110}{00001101}$, $\binom{00100011}{00001100}$, $\binom{00100111}{00001010}$, $\binom{00100101}{00001001}$, $\binom{00100100}{00001110}$, $\binom{00100010}{00011011}$,
$\binom{00100001}{00011111}$, $\binom{00100110}{00011101}$, $\binom{00100011}{00011100}$, $\binom{00100111}{00011010}$, $\binom{00100101}{00011001}$, $\binom{00100100}{00011110}$, $\binom{01000011}{00100001}$, $\binom{01000111}{00100110}$,

$\binom{01000001}{00100100}$, $\binom{01000110}{00100010}$, $\binom{01000010}{00100101}$, $\binom{01000101}{00100011}$, $\binom{01000100}{00100111}$, $\binom{01010001}{00101010}$, $\binom{01010110}{00101001}$, $\binom{01010010}{00101100}$,
$\binom{01010011}{00101110}$, $\binom{01010111}{00101011}$, $\binom{01010101}{00101111}$, $\binom{01010100}{00101101}$, $\binom{01011011}{00110010}$, $\binom{01011111}{00110001}$, $\binom{01011001}{00110101}$, $\binom{01011110}{00110100}$,
$\binom{01011010}{00110111}$, $\binom{01011100}{00110011}$, $\binom{01011101}{00110110}$, $\binom{10000001}{00111011}$, $\binom{10000110}{00111111}$, $\binom{10000011}{00111101}$, $\binom{10000111}{00111100}$, $\binom{10000010}{00111110}$,
$\binom{10000100}{00111001}$, $\binom{10000101}{00111010}$, $\binom{10010001}{00101010}$, $\binom{10010110}{00101001}$, $\binom{10010010}{00101100}$, $\binom{10010011}{00101110}$, $\binom{10010111}{00101011}$, $\binom{10010101}{00101111}$,
$\binom{10010100}{00101101}$, $\binom{10001001}{01000010}$, $\binom{10001110}{01000001}$, $\binom{10001010}{01000100}$, $\binom{10001011}{01000110}$, $\binom{10001111}{01000011}$, $\binom{10001101}{01000111}$, $\binom{10001100}{01000101}$,
$\binom{10010011}{01001001}$, $\binom{10010111}{01001110}$, $\binom{10010001}{01001100}$, $\binom{10010110}{01001010}$, $\binom{10010010}{01001101}$, $\binom{10010100}{01001011}$, $\binom{10010100}{01001111}$, $\binom{10011011}{01001001}$,
$\binom{10011111}{01001110}$, $\binom{10011001}{01001100}$, $\binom{10011110}{01001010}$, $\binom{10011010}{01001101}$, $\binom{10011101}{01001011}$, $\binom{10011100}{01001111}$, $\binom{10011001}{01011010}$, $\binom{10011110}{01011001}$,
$\binom{10011010}{01011100}$, $\binom{10011011}{01011110}$, $\binom{10011111}{01011011}$, $\binom{10011101}{01011111}$, $\binom{10011100}{01011101}$, $\binom{10000001}{01101010}$, $\binom{10000110}{01101001}$, $\binom{10000010}{01101100}$,
$\binom{10000011}{01101110}$, $\binom{10000111}{01101011}$, $\binom{10000101}{01101111}$, $\binom{10000100}{01101101}$, $\binom{10001001}{01100010}$, $\binom{10001110}{01100001}$, $\binom{10001010}{01100100}$, $\binom{10001011}{01100110}$,
$\binom{10001111}{01100011}$, $\binom{10001101}{01100111}$, $\binom{10001100}{01100101}$, $\binom{10001010}{01110001}$, $\binom{10001001}{01110110}$, $\binom{10001110}{01110011}$, $\binom{10001011}{01110111}$, $\binom{10001111}{01110101}$,
$\binom{10001100}{01110010}$, $\binom{10001101}{01110100}$, $\binom{10011001}{01100010}$, $\binom{10011110}{01100001}$, $\binom{10011010}{01100100}$, $\binom{10011011}{01100110}$, $\binom{10011111}{01100011}$, $\binom{10011101}{01100111}$,
$\binom{10011100}{01100101}$, $\binom{10110001}{01010010}$, $\binom{10110110}{01010001}$, $\binom{10110010}{01010100}$, $\binom{10110011}{01010110}$, $\binom{10110111}{01010011}$, $\binom{10110101}{01010111}$, $\binom{10110100}{01010101}$,
$\binom{10101001}{01101011}$, $\binom{10101110}{01101111}$, $\binom{10101011}{01101101}$, $\binom{10101111}{01101100}$, $\binom{10101010}{01101110}$, $\binom{10101100}{01101001}$, $\binom{10101101}{01101010}$, $\binom{10100001}{01110010}$,
$\binom{10100110}{01110001}$, $\binom{10100010}{01110100}$, $\binom{10100011}{01110110}$, $\binom{10100111}{01110011}$, $\binom{10100101}{01110111}$, $\binom{10100100}{01110101}$, $\binom{10100001}{01111010}$, $\binom{10100110}{01111001}$,
$\binom{10100010}{01111100}$, $\binom{10100011}{01111110}$, $\binom{10100111}{01111011}$, $\binom{10100101}{01111111}$, $\binom{10100100}{01111101}$, $\binom{10101001}{01110010}$, $\binom{10101110}{01110001}$, $\binom{10101010}{01110100}$,
$\binom{10101011}{01110110}$, $\binom{10101111}{01110011}$, $\binom{10101101}{01110111}$, $\binom{10101100}{01110101}$, $\binom{10110001}{01111011}$, $\binom{10110110}{01111111}$, $\binom{10110011}{01111101}$, $\binom{10110111}{01111100}$,
$\binom{10110010}{01111110}$, $\binom{10110100}{01111001}$, $\binom{10110101}{01111010}$, $\binom{11000001}{00110010}$, $\binom{11000110}{00110001}$, $\binom{11000010}{00110100}$, $\binom{11000011}{00110110}$, $\binom{11000111}{00110011}$,
$\binom{11000101}{00110111}$, $\binom{11000100}{00110101}$, $\binom{11001001}{00111010}$, $\binom{11001110}{00111001}$, $\binom{11001010}{00111100}$, $\binom{11001011}{00111110}$, $\binom{11001111}{00111011}$, $\binom{11001101}{00111111}$,
$\binom{11001100}{00111101}$, $\binom{11010010}{00110001}$, $\binom{11010001}{00110110}$, $\binom{11010110}{00110011}$, $\binom{11010011}{00110111}$, $\binom{11010111}{00110101}$, $\binom{11010100}{00110010}$, $\binom{11010101}{00110100}$,
$\binom{01001000}{00100000}$, $\binom{01001000}{00101000}$, $\binom{01000000}{00110000}$, $\binom{01000000}{00111000}$, $\binom{01110000}{00001000}$, $\binom{10000000}{00011000}$, $\binom{10001000}{00101000}$, $\binom{10010000}{00100000}$,
$\binom{10101000}{00011000}$, $\binom{10001000}{01010000}$, $\binom{10011000}{01011000}$, $\binom{10001000}{01100000}$, $\binom{10000000}{01110000}$, $\binom{10111000}{01101000}$, $\binom{10111000}{01111000}$, $\binom{11001000}{00010000}$,
$\binom{11001000}{00100000}$, $\binom{11010000}{00101000}$, $\binom{11010000}{00110000}$, $\binom{11011000}{00111000}$;

$s = 50$: $\binom{00100010}{00011011}$, $\binom{00100001}{00011111}$, $\binom{00100110}{00011101}$, $\binom{00100011}{00011100}$, $\binom{00100111}{00011010}$, $\binom{00100101}{00011001}$, $\binom{00100100}{00011110}$,
$\binom{00101010}{00010001}$, $\binom{00101001}{00010110}$, $\binom{00101110}{00010011}$, $\binom{00101011}{00010111}$, $\binom{00101111}{00010101}$, $\binom{00101100}{00010010}$, $\binom{00101101}{00010100}$, $\binom{01000010}{00001001}$,
$\binom{01000001}{00001110}$, $\binom{01000110}{00001011}$, $\binom{01000011}{00001111}$, $\binom{01000111}{00001101}$, $\binom{01000100}{00001010}$, $\binom{01000101}{00001100}$, $\binom{01001010}{00111001}$, $\binom{01001001}{00111110}$,
$\binom{01001110}{00111011}$, $\binom{01001011}{00111111}$, $\binom{01001111}{00111101}$, $\binom{01001100}{00111010}$, $\binom{01001101}{00111100}$, $\binom{01010001}{00101010}$, $\binom{01010110}{00101001}$, $\binom{01010010}{00101100}$,
$\binom{01010011}{00101110}$, $\binom{01010111}{00101011}$, $\binom{01010101}{00101111}$, $\binom{01010100}{00101101}$, $\binom{01100001}{00001010}$, $\binom{01100110}{00001001}$, $\binom{01100010}{00001100}$, $\binom{01100011}{00001110}$,
$\binom{01100111}{00001011}$, $\binom{01100101}{00001111}$, $\binom{01100100}{00001101}$, $\binom{10001001}{00011010}$, $\binom{10001110}{00011001}$, $\binom{10001010}{00011100}$, $\binom{10001011}{00011110}$, $\binom{10001111}{00011011}$,
$\binom{10001101}{00011111}$, $\binom{10001100}{00011101}$, $\binom{10010001}{00001010}$, $\binom{10010110}{00001001}$, $\binom{10010010}{00001100}$, $\binom{10010011}{00001110}$, $\binom{10010111}{00001011}$, $\binom{10010101}{00001111}$,
$\binom{10010100}{00001101}$, $\binom{10011001}{00101010}$, $\binom{10011110}{00101001}$, $\binom{10011010}{00101100}$, $\binom{10011011}{00101110}$, $\binom{10011111}{00101011}$, $\binom{10011101}{00101111}$, $\binom{10011100}{00101101}$,
$\binom{10101001}{00010010}$, $\binom{10101110}{00010001}$, $\binom{10101010}{00010100}$, $\binom{10101011}{00010110}$, $\binom{10101111}{00010011}$, $\binom{10101101}{00010111}$, $\binom{10101100}{00010101}$, $\binom{10000010}{01000011}$,
$\binom{10000001}{01000111}$, $\binom{10000110}{01000101}$, $\binom{10000011}{01000100}$, $\binom{10000111}{01000010}$, $\binom{10000101}{01000001}$, $\binom{10000100}{01000110}$, $\binom{10001010}{01001011}$, $\binom{10001001}{01001111}$,
$\binom{10001110}{01001101}$, $\binom{10001011}{01001100}$, $\binom{10001111}{01001010}$, $\binom{10001101}{01001001}$, $\binom{10001100}{01001110}$, $\binom{10000010}{01011001}$, $\binom{10000001}{01011110}$, $\binom{10000110}{01011011}$,
$\binom{10000011}{01011111}$, $\binom{10000111}{01011101}$, $\binom{10000100}{01011010}$, $\binom{10000101}{01011100}$, $\binom{10010001}{01101010}$, $\binom{10010110}{01101001}$, $\binom{10010010}{01101100}$, $\binom{10010011}{01101110}$,
$\binom{10010111}{01101011}$, $\binom{10010101}{01101111}$, $\binom{10010100}{01101101}$, $\binom{10100001}{01011010}$, $\binom{10100110}{01011001}$, $\binom{10100010}{01011100}$, $\binom{10100011}{01011110}$, $\binom{10100111}{01011011}$,
$\binom{10100101}{01011111}$, $\binom{10100100}{01011101}$, $\binom{10110001}{01010010}$, $\binom{10110110}{01010001}$, $\binom{10110010}{01010100}$, $\binom{10110011}{01010110}$, $\binom{10110111}{01010011}$, $\binom{10110101}{01010111}$,
$\binom{10110100}{01010101}$, $\binom{10111001}{01011010}$, $\binom{10111110}{01011001}$, $\binom{10111010}{01011100}$, $\binom{10111011}{01011110}$, $\binom{10111111}{01011011}$, $\binom{10111101}{01011111}$, $\binom{10111100}{01011101}$,
$\binom{10100001}{01101010}$, $\binom{10100110}{01101001}$, $\binom{10100010}{01101100}$, $\binom{10100011}{01101110}$, $\binom{10100111}{01101011}$, $\binom{10100101}{01101111}$, $\binom{10100100}{01101101}$, $\binom{10100001}{01111010}$,

$\binom{10100110}{01111001}, \binom{10100010}{01111100}, \binom{10100011}{01111110}, \binom{10100111}{01111011}, \binom{10100101}{01111111}, \binom{10100100}{01111101}, \binom{10101001}{01111010}, \binom{10101110}{01111001},$
$\binom{10101010}{01111100}, \binom{10101011}{01111110}, \binom{10101111}{01111011}, \binom{10101101}{01111111}, \binom{10101100}{01111101}, \binom{10110001}{01100010}, \binom{10110110}{01100001}, \binom{10110010}{01100100},$
$\binom{10110011}{01100110}, \binom{10110111}{01100011}, \binom{10110101}{01100111}, \binom{10110100}{01100101}, \binom{11001010}{00100001}, \binom{11001001}{00100110}, \binom{11001110}{00100011}, \binom{11001011}{00100111},$
$\binom{11001111}{00100101}, \binom{11001100}{00100010}, \binom{11001101}{00100100}, \binom{11000001}{00110010}, \binom{11000110}{00110001}, \binom{11000010}{00110100}, \binom{11000011}{00110110}, \binom{11000111}{00110011},$
$\binom{11000101}{00110111}, \binom{11000100}{00110101}, \binom{11010010}{00110001}, \binom{11010001}{00110110}, \binom{11010110}{00110011}, \binom{11010011}{00110111}, \binom{11010111}{00110101}, \binom{11010100}{00110010},$
$\binom{11010101}{00110100}, \binom{11101001}{00011010}, \binom{11101110}{00011001}, \binom{11101010}{00011100}, \binom{11101011}{00011110}, \binom{11101111}{00011011}, \binom{11101101}{00011111}, \binom{11101100}{00011101},$
$\binom{00010000}{00001000}, \binom{00110000}{00001000}, \binom{01001000}{00100000}, \binom{01001000}{00110000}, \binom{01011000}{00111000}, \binom{01101000}{00010000}, \binom{10000000}{00011000}, \binom{10010000}{00101000},$
$\binom{10011000}{00101000}, \binom{10100000}{00011000}, \binom{10000000}{01011000}, \binom{10001000}{01010000}, \binom{10001000}{01011000}, \binom{10010000}{01110000}, \binom{10010000}{01111000}, \binom{10100000}{01000000},$
$\binom{10100000}{01001000}, \binom{10101000}{01010000}, \binom{10110000}{01000000}, \binom{10101000}{01101000}, \binom{10110000}{01100000}, \binom{11000000}{00001000}, \binom{11000000}{00100000}, \binom{11001000}{00111000},$
$\binom{11010000}{00101000}$.

With this, all remaining constructions can be obtained using Theorem 3.8 and Lemma 2.11. □

In Table 2 we state the known bounds for $n_4(s)$ when $s \leq 60$. Lower bounds based on quaternary linear codes are stated in columns headed with "L". Upper bounds, based on [11] for $s \leq 4$ and on binary linear codes for $s > 4$, i.e. the strong coding upper bound, are stated in columns headed with "U". Actually, for $s \geq 5$ the weak coding upper bound is sufficient in all but three cases, which we indicate in bold face and discuss in detail in Subsection 4.1. Values of improved constructions are given in columns headed with "I". We remark that we have $n_4(s) = n_4(s-21) + 85$ for $n > 60$. For $s > 60$ there are improvements over the linear case iff s is congruent to 2, 3, 7, or 8 modulo 21. Generator matrices of the improvements are given in the proof of Theorem 4.2. We observe that $n_4(44) \geq n_4(23) + n_4(21)$ and $n_4(45) \geq n_4(24) + n_4(21)$ are attained with equality.

4.1 Non-existence of binary linear codes related to quaternary additive codes

In this subsection we want to evaluate the strong coding upper bound for $n_4(15)$, $n_4(28)$, and $n_4(29)$, i.e. the three values in Table 2 which have a bold entry in the "U"-column.[2] It will turn out that the weak coding upper bound is improved by two in those three cases. The strong coding upper bound is based on Lemma 2.3, i.e. we have to show the non-existence of certain binary linear codes taking an upper bound on the maximum weight into account. To this end we denote an $[n,k]_2$ code with non-zero weights of the codewords contained in $W \subsetneq \mathbb{N}$ as an $[n,k,W]_2$ code and use the software package `LinCode`[24] for

[2] For $n_4(3)$ and $n_4(4)$ we state that there exist 217 non-isomorphic even $[18,7,\{6,\ldots,12\}]_2$ and more than 10^6 non-isomorphic even $[33,7,\{14,\ldots,22\}]_2$ codes, i.e. the strong coding upper bound does not imply the exact values.

s	L	I	U	s	L	I	U	s	L	I	U
1	–		–	21	85		85	41	165		165
2	–		–	22	86		86	42	170		170
3	5		5	23	87	89	89	43	171		171
4	10		10	24	92	94	94	44	172	174	174
5	17		17	25	97		97	45	177	179	179
6	18		18	26	102		102	46	182		182
7	23		23	27	103	107	107	47	187		187
8	28		28	28	108		**108**	48	192		192
9	31	33	33	29	113		**113**	49	193	195	195
10	34	36	36	30	118		118	50	198	200	200
11	39	40	40	31	123		123	51	203		203
12	44		44	32	128		128	52	208		208
13	49		49	33	129		129	53	213		213
14	50	54	54	34	134		134	54	214		214
15	55		**55**	35	139		139	55	219		219
16	64		64	36	144		144	56	224		224
17	65		65	37	149		149	57	229		229
18	70		70	38	150		150	58	234		234
19	75		75	39	155		155	59	235		235
20	80		80	40	160		160	60	240		240

Table 2: Bounds for $n_4(s)$.

the exhaustive enumeration of linear codes. More precisely, we will recursively construct $[n,k,W]_2$ codes from $[n',k-1,W]_2$ codes for some suitable set $W \subsetneq \mathbb{N}$ of possible non-zero weights. Initially we will start with the unique $[d,1,d]_2$ code for the desired minimum distance d. Given some $[n',k-1,W]_2$ code C we check that they cannot be extended to $[n',k,W]_2$ codes. Let the parameters of the desired final code $\tilde{C}$ be denoted by $[\tilde{n},\tilde{k},W]_2$. The residual code of $\tilde{C}$ with respect to the support of C is an $[\tilde{n}-n',\tilde{k}-k+1,d']_2$. If $\hat{d}$ is an upper bound on d', then it suffices to consider the case $n \leq n+d'$. In Table 3 we state the counts for the intermediate codes aiming at a $[330,8,164]_2$ code with maximum weight at most 220. Since no $[330,8,165]_2$ code exists, we can assume the existence of a codeword of weight 164. The minimum distance of a residual $[166,7]_2$ code is at most 82, so that it suffices to consider $[\leq 246,2,W]_2$ codes. For a $[246,2,W]_2$ code the residual code is a $[84,6]_2$ code with minimum distance at most 41, for

n	164	246	287	308	319	325	328	330
k	1	2	3	4	5	6	7	8
#	1	1	1	1	3	34	69	0

Table 3: Number of $[n, k, \{164, 168, \ldots, 220\}\backslash\{196\}]_2$ codes which are recursively obtainable.

n	172	258	301	323	334	340	343	345
k	1	2	3	4	5	6	7	8
#	1	1	1	1	1	2	3	0

Table 4: Number of $[n, k, \{172, 176, 184, 188, 192, 200, 216, 220, 224, 228\}]_2$ codes which are recursively obtainable.

a $[287, 3, W]_2$ code the residual code is a $[43, 5]_2$ code with minimum distance at most 21, for a $[308, 4, W]_2$ code the residual code is a $[22, 4]_2$ code with minimum distance at most 11, for a $[319, 5, W]_2$ code the residual code is a $[11, 3]_2$ code with minimum distance at most 6, and for a $[325, 6, W]_2$ code the residual code is a $[5, 2]_2$ code with minimum distance at most 2. Note that the possible minimum lengths of $[n, k, 164]_2$ codes are indeed given by 164, 246, 287, 308, 319, 325, 328, and 330 for $1 \leq k \leq 8$, all given by the Griesmer bound.

While $[330, 8, 164]_2$ and $[345, 8, 172]_2$ codes can be easily constructed from the Solom–Stiffler construction, the resulting maximum weights violate Lemma 2.3. In Lemma 4.3 and Lemma 4.5 we show that this indeed the case in general.

Lemma 4.3. *No* $[330, 8, 164]_2$ *code with maximum weight at most* 220 *exists.*

Proof. Assume that C is a $[330, 8, 164]_2$ code with maximum weight at most 220. From Lemma 2.8 we conclude that C is 4-divisible. From Lemma 2.7 and the non-existence of the corresponding residual codes we conclude that C does not have a codeword with weight 196. So, consider the set $W = \{164, 168, \ldots, 220\}\backslash\{196\}$ of possible non-zero weights. Using `LinCode`[24] we have iteratively constructed $[n, k, W]_2$ codes as stated in Table 3. □

Corollary 4.4. $n_4(28) \leq 109$.

Lemma 4.5. *No* $[345, 8, 172]_2$ *code with maximum weight at most* 230 *exists.*

Proof. Assume that C is $[345, 8, 172]_2$ code with maximum weight at most 228. From Lemma 2.8 we conclude that C is 4-divisible. From Lemma 2.7 and the non-existence of the corresponding residual codes we conclude that C does not have a codeword with a weight in $\{180, 196, 204, 208, 212, 230\}$. So, consider the set $W := \{172, 176, 184, 188, 192, 200, 216, 220, 224, 228\}$ of possible non-zero weights. Using `LinCode`[24] we have iteratively constructed $[n, k, W]_2$ codes as stated in Table 4. □

Corollary 4.6. $n_4(29) \leq 114$.

While the computation underlying Lemma 4.5 took just a few minutes, for Lemma 4.3 forty-four hours were needed. In principle the non-existence of an even $[171, 8, 84]_2$ code with maximum weight at most 114, i.e., the binary code corresponding to a $(57, 8, 15)$ system, can be obtained using the same computational approach. However, the required computation time would be significantly large, which is partially due to the fact that the cannot deduce 4-divisibility[3] from Lemma 2.7 and, more importantly, since $g(8, 84) = 170 < 171$. The known optimal $[171, 8, 84]_2$ code contains a codeword of weight 128 and two codewords of weight 100.[4]

Slightly enhancing our computational approach, we show the non-existence of a $[168, 8, 82]_2$ code with maximum weight 112. By Lemma 2.3 no $(56, 8, 15)$ system exists. Thus, also no $(55, 8, 15)$ system can exist. We can also deduce the slightly stronger result of the non-existence of a $[171, 8, 84]_2$ code C with maximum weight at most 114 by using a parity bit argument and by deducing the existence of a full line in the support of C.

Definition 4.7. Let C be an $[n, k]_2$ code and $\mathcal{M}_k$ be its corresponding multiset of points in $\mathrm{PG}(k-1, 2)$. We recursively choose points P_i and construct $\mathcal{M}_i$ from $\mathcal{M}_{i+1}$ by projection through P_i. We call $\left(|\mathcal{M}_k(P_k)|, \ldots, |\mathcal{M}_1(P_1)|\right)$ the extensions vector with respect to the points $P_k, \ldots, P_1$. The lexicographic maximum over all choices is called the (canonical) extension vector of $\mathcal{M}$.

Clearly, the points $P_k, \ldots, P_1$ of the canonical extension vector have to be endpoints of $\mathcal{M}_i$, i.e. points with the maximum multiplicity.

[3] 4-divisibility can be concluded nevertheless: By a residual code argument the only possible weights not divisible by four can be reduced to $\{90, 106, 110\}$. From Proposition 2.10 with $a = 1$ we can conclude $T \in \{192, 256\}$ using the ILP method and the existence of a 4-divisible subcode with dimension at least 7. Applying the ILP method for the corresponding split weight enumerator, see e.g. [14, 15] for details, those three weights can be excluded relatively easily.

[4] Note that the residual code of a codeword of weight 100 is a unique $[71, 7, 34]_2$ code, see e.g. [25], and especially contains a codeword of weight 64. This is impossible given a maximum weight of 114 in the original code.

Remark 4.8. For a multiset of points $\mathcal{M}$ in $\mathrm{PG}(k-1,q)$ the maximum multiplicity of an i-dimensional subspace S_i is usually denoted by $\gamma_i(\mathcal{M})$. I.e. $\gamma_1(\mathcal{M})$ is the maximum point multiplicity and $\gamma_k(\mathcal{M}) = |\mathcal{M}|$. As a convention we set $\gamma_0(\mathcal{M}) = 0$. If $\mathcal{M}$ is spanning and C denotes the linear code corresponding to $\mathcal{M}$, then $|\mathcal{M}| - \gamma_i(\mathcal{M})$ equals the maximum possible minimum support weight of q $(k-i)$-dimensional subcodes of C. Note that

$$(\gamma_1(\mathcal{M}) - \gamma_0(\mathcal{M}), \ldots, \gamma_k(\mathcal{M}) - \gamma_{k-1}(\mathcal{M}))$$

might be lexicographically larger than the canonical extension vector of $\mathcal{M}$ since the subspaces $S_i = \langle P_k, \ldots, P_{k-i+1} \rangle$ have to form a chain $S_1 \subsetneq \cdots \subsetneq S_k$ for the latter case

Lemma 4.9. *No* $[168, 8, 82]_2$ *code with maximum weight at most* 112 *exists.*

Proof. Assume that C' is a $[168, 8, 82]_2$ code with maximum weight at most 112. Let C arise from C' by shortening one coordinate and adding a parity bit. So, C is an even $[168, 8, 82]_2$ code and we consider a residual code w.r.t. a codeword of weight w.

- For $w = 82$ the residual code is an $[86, 7, 41]_2$ code with non-zero weights in $\{41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 57, 58, 61, 62, 63, 64\}$.
- For $w = 90$ the residual code is a $[78, 7, 37]_2$ code with non-zero weights in $\{37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 53, 54, 55, 56, 61, 62, 63, 64\}$.
- For $w = 94$ the residual code is a $[74, 7, 35]_2$ code with non-zero weights in $\{35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 47, 48, 51, 52, 55, 56, 59, 60, 63, 64, 67, 68\}$.
- For $w = 98$ the residual code is a $[70, 7, 33]_2$ code with non-zero weights in $\{33, 34, 35, 36, 37, 38, 63, 64\}$.
- For $w = 110$ the residual code is a $[58, 7, 27]_2$ code with non-zero weights in $\{27, 28, 31, 32, 35, 36, 39, 40, 43, 44, 47, 48, 51, 52\}$.
- For $w = 112$ the residual code is a $[56, 7, 26]_2$ code with non-zero weights in $\{26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56\}$.

Here we use the classification of optimal $[n, \leq 7]_2$ codes from [25] and shortening if the minimum distance is odd. Since no $[165, 7, 82]_2$ code exists the maximum column multiplicity of C is at most 2. We have verified by exhaustive enumeration that no 2-divisible $[166, 7, \{82, \ldots, 112\}]_2$ code exists. (The known optimal $[166, 7, 82]_2$ code contains a codeword of weight 128.) Thus, C has to be projective as do all residual codes of C.

Let $W := \{82, 84, \ldots, 112\}$. Starting from an $[82, 1, W]_2$ code we construct extensions to $[146, 2, W]_2$ codes and then recursively consider all extensions that

may lead to a $[168, 8, W]_2$ code. During the generation with `LinCode` we remove codes that contain a residual code with a forbidden weight, as specified above. No $[168, 8, W]_2$ code was reached, see Table 5 for the counting details. So, in the following we can assume that each residual code of a codeword of weight 82 does not contain a codeword of weight 64 and we continue to exclude further weights. To this end we refine the extension strategy a bit. Let $c, c' \in C$ be two different codewords in C where c has weight 82. Let $\tilde{c}$ be the codeword in the residual code of c that corresponds to c', i.e. the residual code with respect to the support of c and c' is a projective $[168 - \mathrm{wt}(c) - \mathrm{wt}(\tilde{c}), 6, d]_2$ code C'. By enumerating all even $[\leq 168, 3, \{82, \ldots, 112\}]_2$ codes we have verified that for e.g. $\mathrm{wt}(\tilde{c}) \in \{45, 49, 57, 58, 61, 62, 63, 64\}$, C' has to be a distance optimal code. Counts per canonical extension vector are given as follows:

- $\mathrm{wt}(\tilde{c}) = 64$, $[22, 6, 9]_2$: $(1, 2, 2, 3, 5, 9) \to 184$, $(1, 1, 2, 4, 5, 9) \to 1$;
- $\mathrm{wt}(\tilde{c}) = 63$, $[23, 6, 10]_2$: $(1, 2, 2, 3, 5, 10) \to 21$;
- $\mathrm{wt}(\tilde{c}) = 62$, $[24, 6, 10]_2$: $(1, 2, 3, 3, 5, 10) \to 2064$, $(1, 2, 2, 4, 5, 10) \to 74$, $(1, 2, 2, 3, 6, 10) \to 11$, $(1, 1, 2, 4, 6, 10) \to 2$;
- $\mathrm{wt}(\tilde{c}) = 61$, $[25, 6, 11]_2$: $(1, 2, 2, 3, 6, 11) \to 3$;
- $\mathrm{wt}(\tilde{c}) = 58$, $[28, 6, 12]_2$: $(1, 2, 3, 4, 6, 12) \to 4182$, $(1, 2, 4, 3, 6, 12) \to 1588$, $(1, 2, 2, 4, 7, 12) \to 1$;
- $\mathrm{wt}(\tilde{c}) = 57$, $[29, 6, 13]_2$: $(1, 1, 2, 4, 8, 13) \to 1$;
- $\mathrm{wt}(\tilde{c}) = 49$, $[37, 6, 17]_2$: $(1, 2, 3, 5, 9, 17) \to 2$;
- $\mathrm{wt}(\tilde{c}) = 45$, $[41, 6, 19]_2$: $(1, 2, 4, 5, 10, 19) \to 10$, $(1, 2, 3, 6, 10, 19) \to 7$.

With this, we have computationally excluded the cases $\mathrm{wt}(\tilde{c}) \in \{58, 61, 63, 64\}$ for a residual code of a codeword of weight 82. We have enumerated all 15 projective $[86, 7, \{41, \ldots, 54, 57, 62\}]_2$ codes. It turns out that weights 53, 54, 57 do not occur and that weight 62 always occurs. Moreover, the maximum frequencies for the larger weights are given by $62^1 52^1 51^1 50^2 49^4 48^3 47^6$. With these additional restrictions at hand, we computationally exclude the case $\mathrm{wt}(\tilde{c}) = 82$, so that C cannot contain a codeword c of weight 82 – contradiction. The overall computation took 10 hours. □

From Lemma 2.3, Lemma 4.9, and the existence of an $[55, 4, 40]_4$ code[5] we directly conclude:

Theorem 4.10. $n(15) = 55$.

[5] Adding an arbitrary line or a double-point to a $(54, 8, 14)$ system also yields $(55, 8, 15)$ systems.

n	146	155	160	163	164	166	167	168
k	2	3	4	5	5	6	7	8
#	3	2	2	2	43	74	0	0

Table 5: Number of $[n, k, \{82, 84, \ldots, 112\}]_2$ codes which are recursively obtainable from a $[146, 2, W]_2$ code containing a codeword of weight 82.

Remark 4.11. In the computational approach of the proof of Lemma 4.9 we have a lot of choices. We may directly enumerate the 257 projective $[86, 7, 41]_2$ codes to conclude the forbidden weights for the residual code without using the results from [25]. While the additional assumption of a projective code does not give more forbidden weights in our example, we might use the maximum weight frequencies $64^1 63^1 62^1 61^1 58^1 57^2 54^2 53^2 52^2 51^2 50^2 49^4 48^{10}$ directly from the start. We mention that one might also deduce more refined conditions like that the number of residual codewords of weight at least 58 is at most 1 or to use a lexicographic comparison with all possible weight enumerators.

There is some similarity to the linear programming method based on the split weight enumerator as used in [15]. We also remove certain partitions from the search tree. However, we do not use linear programming but exhaustive enumeration and information on residual codes. Instead of forcing the existence of codewords of a certain partition by linear programming we use available information of the possible minimum distances of (residual) codes.

The use of minimum distance bounds for residual codes in the generation tree comes with some subtleties. If we aim at a $[168, 8, \{82, 84, \ldots, 112\}]_2$ code starting from the $[82, 1, 82]_2$ code without using weight information for the residual codes, then the partial counting information is given in Table 6. However, the number of $[145, 3, \{82, 84, \ldots, 112\}]_2$, $[156, 4, \{82, 84, \ldots, 112\}]_2$, and $[162, 2, \{82, 84, \ldots, 112\}]_2$ codes is given by 4, 30, and 1524, respectively. The missing 3-dimensional code is geometrically obtained by removing a double-point from a plane with multiplicity 21.[6] Note that this code does not contain a $[123, 2, 82]_2$ code as a subcode, but only a $[124, 2, 82]_2$ code.

Lemma 4.12. *No* $[327, 8, 162]_2$ *code with maximum weight at most* 218 *exists.*

Proof. Assume that C is a $[327, 8, 162]_2$ code with maximum weight at most 218. From Lemma 2.8 we conclude that C is 2-divisible. From Lemma 2.7 and the non-

[6] All four non-isomorphic even $[145, 3, 82]_2$ codes can geometrically be obtained by removing three lines from a 22-fold plane.

n	82	123	144	145	155	156	161	162	...	168
k	1	2	3	3	4	4	5	5	...	8
#	1	1	1	3	3	28	23	1509	...	0

Table 6: Number of $[n, k, \{82, 84, \ldots, 112\}]_2$ codes which are recursively obtainable from a $[82, 1, 82]_2$ code.

existence of the corresponding residual codes we conclude that C does not have a codeword with weight 194. So, consider the set $W = \{162, 164, \ldots, 218\}\backslash\{194\}$ of possible non-zero weights. For a codeword of weight 162 the residual code C' is a $[165, 7, 81]_2$ Griesmer code. We have enumerated all 53 even $[166, 7, 82]_2$ codes.[7] Since their non-zero weights are all contained in $\{82, \ldots, 96, 114, \ldots, 124, 128\}$ and the maximum weight is at least 114, the non-zero weights of C' are all contained in $\{81, \ldots, 96, 113, \ldots, 124, 127, 128\}$ and the maximum weight is at least 113. Starting from the unique $[162, 1, W]_2$ code we constructed all $[n, 2, W]_2$ codes with $n \in \{275, \ldots, 290\}\backslash\{287, 288\}$. There are no $[n', 3, W]_2$ code which are obtainable from these codes and satisfy the mentioned constraints for the residual codes of codewords of weight 162. So, C cannot contain a codeword of weight 162 – contradiction. The overall computation took less than two hours. □

From Lemma 2.3, Lemma 4.12, and the existence of an $[108, 4, 80]_4$ code[8] we directly conclude:

Theorem 4.13. $n_4(28) = 108$.

Lemma 4.14. *No* $[342, 8, 170]_2$ *code with maximum weight at most* 228 *exists.*

Proof. Assume that C is a $[342, 8, 170]_2$ code with maximum weight at most 228. From Lemma 2.8 we conclude that C is 2-divisible. From Lemma 2.7 and the non-existence of the corresponding residual codes we conclude that C does not have a codeword with a weight in $\{178, 194, 202, 206, 208, 210\}$. So, consider the set $W = \{170, 172, \ldots, 228\}\backslash\{178, 194, 202, 206, 208, 210\}$ of possible non-zero weights. For a codeword of weight 170 the residual code C' is a $[172, 7, 85]_2$ code. We have enumerated all five even $[173, 7, 86]_2$ codes. Since their non-zero weights are all contained in $\{86, 88, 94, 96, 118, 120, 126, 128\}$ and

[7] Using the projective dual transform with $\alpha = \frac{1}{2}$, $\beta = -41$, and $m = 7$ these codes also correspond to $[105, 7, \{32, 48, 64\}]_2$ codes, see [26].

[8] Adding an arbitrary line or a double-point to a $(107, 8, 27)$ system also yields $(108, 8, 28)$ systems.

the maximum weight is at least 118, the non-zero weights of C' are all contained in $\{85, \ldots, 88, 93, \ldots, 96, 117, \ldots, 120, 125, \ldots, 128\}$ and the maximum weight is at least 117. Starting from the unique $[170, 1, W]_2$ code we constructed all $[n, 2, W]_2$ codes with $n \in \{287, \ldots, 298\} \backslash \{291, \ldots, 294\}$. There are no $[n', 3, W]_2$ code which are obtainable from these codes and satisfy the mentioned constraints for the residual codes of codewords of weight 170. So, C cannot contain a codeword of weight 170 – contradiction. The overall computation took less than five minutes. □

From Lemma 2.3, Lemma 4.14, and the existence of an $[113, 4, 84]_4$ code we directly conclude:

Theorem 4.15. $n_4(29) = 113$.

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