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Locally Two-weight Property for Linear Codes and Its Application

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Abstract

A q-ary linear code is an $[n, k, d]_q$ code, which is a linear code of length n, dimension k and minimum weight d over $\mathbb{F}_q$, the field of order q. A fundamental problem in coding theory is to find $n_q(k, d)$, the minimum length n for which an $[n, k, d]_q$ code exists for given k, d and q. We introduce a new notion "e-locally 2-weight $(\bmod\ q)$" for linear codes over $\mathbb{F}_q$ and we give a necessary condition for the property. As an application, we prove the non-existence of some $[n, 4, d]_9$ codes with $d \equiv -1 \pmod 9$, which determines $n_9(4, d)$ for some d.

Keywords: Linear Codes, Two-weight, Non-existence, Geometric Method

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1 Introduction

We denote by $\mathbb{F}_q$ the field of q elements. Let $\mathbb{F}_q^n$ be the vector space of n-tuples over $\mathbb{F}_q$. A k-dimensional subspace $\mathcal{C}$ of $\mathbb{F}_q^n$ is called a linear code of length n and dimension k, or an $[n, k]_q$ code. $\mathcal{C}$ is also called an $[n, k, d]_q$ code if the minimum Hamming weight $\min\{wt(c) \mid c \in \mathcal{C}, c \neq (0, \dots, 0)\}$ is d, where $wt(c)$ is the number of non-zero entries in the vector c. We only consider linear codes having no coordinate which is identically zero.

A fundamental problem in coding theory is to find $n_q(k, d)$, the minimum length n for which an $[n, k, d]_q$ code exists [1, 2]. The length n of an $[n, k, d]_q$ code satisfies the following inequality called the *Griesmer bound* [3, 4]:

$$n \geq g_q(k, d) = \sum_{i=0}^{k-1} \left\lceil d/q^i \right\rceil ,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. The values of $n_q(k, d)$ are determined for all d only for some small values of q and k. For $k = 3$, $n_q(3, d)$ is known for all d for $q \leq 9$. See [5] for the updated linear codes bound. As for the updated tables of $n_q(k, d)$ for some small q and k, see [6]. It is known that Griesmer codes do exist if d is large enough for given q and k [1]. But the problem to find all d such that $[g_q(k, d), k, d]_q$ codes exist is still open. For example, the following results are known for $n_9(4, d)$.

Theorem 1.1 ([7]).

(a) $n_9(4, d) = g_9(4, d) + 1$ *for* $d = 585, 810$,

(b) $n_9(4, d) = g_9(4, d)$ *or* $g_9(4, d) + 1$ *for* $d = 584, 809$,

(c) $n_9(4, d) \geq g_9(4, d) + 1$ *for* $d = 198$.

It is not known if a $[g_9(4, d), 4, d]_9$ code exist or not for $d = 197$. See also [8] for the recent progress on $n_9(4, d)$. We prove the non-existence of $[g_9(4, d), 4, d]_9$ codes for $d = 197, 584, 809$ as an application of our main result, giving the following.

Theorem 1.2.

(a) $n_9(4, d) = g_9(4, d) + 1$ *for* $d = 584, 809$,

(b) $n_9(4, d) \geq g_9(4, d) + 1$ *for* $d = 197$.

The weight distribution of $\mathcal{C}$ is the list of positive integers A_i, where A_i is the number of codewords of weight i, $0 \le i \le n$. The weight distribution with $(A_0, A_d, \ldots) = (1, \alpha, \ldots)$ is expressed as $0^1 d^\alpha \cdots$. A linear code $\mathcal{C}$ over $\mathbb{F}_q$ is *w-weight* $(mod\ q)$ if there exists a w-set $W = \{i_1, \cdots, i_w\} \subset \mathbb{Z}_q = \{0, 1, \cdots, q-1\}$ such that $A_i > 0$ implies $i \equiv i_j \pmod q$ for some $i_j \in W$. For example, the well-known Golay $[11, 6, 5]_3$ code has weight distribution $0^1 5^{132} 6^{132} 8^{330} 9^{110} 11^{24}$, which is 2-weight (mod 3).

The code obtained by deleting the same coordinate from each codeword of an $[n, k, d]_q$ code $\mathcal{C}$ is called a *punctured code* of $\mathcal{C}$. If there exists an $[n+1, k, d+1]_q$ code $\mathcal{C}'$ which gives $\mathcal{C}$ as a punctured code, $\mathcal{C}$ is called *extendable* and $\mathcal{C}'$ is an *extension* of $\mathcal{C}$. The Golay $[11, 6, 5]_3$ code is extendable by the following result proved by Hill and Lizak (1995).

Theorem 1.3 ([9, 10])**.** *Every $[n, k, d]_q$ code with gcd$(d, q) = 1$ whose weights of codewords are congruent to 0 or $d \pmod q$ is extendable.*

Assume $\gcd(d, q) = 1$. Let $\mathcal{C}$ be an $[n, k, d]_q$ code with generator matrix G. Let V_m be an m-dimensional subspace of $\mathbb{F}_q^k$. We say that $\mathcal{C}$ is *locally 2-weight* $\pmod q$ *at V_m* if

$$wt(vG) \equiv 0 \text{ or } d \pmod q \tag{1}$$

for all $v \in V_m$. We also say that $\mathcal{C}$ is *e-locally 2-weight* $\pmod q$ *at V_m* if (1) holds for all $v \in V_m$ except for $e(q-1)$ vectors.

Example 1.1. Let $\mathcal{C}_1$ be the $[23, 4, 14]_3$ code with generator matrix

$$G_1 = \begin{bmatrix} 10001111111111111000000 \\ 01002222211111000111100 \\ 00102210022100210112210 \\ 00011022020012201211011 \end{bmatrix}.$$

Then, $\mathcal{C}_1$ has weight distribution $0^1 14^{22} 15^{24} 16^{18} 17^{12} 18^2 23^2$. Take a 2-dimensional subspace of $\mathbb{F}_3^4$ as

$$V_1 = \{(0, 0, 0, 0), v_1, 2v_1, v_2, 2v_2, v_3, 2v_3, v_4, 2v_4\},$$

where $v_1 = (1, 0, 0, 0), v_2 = (1, 0, 0, 1), v_3 = (1, 0, 0, 2), v_4 = (0, 0, 0, 1)$. Since $wt(v_1G) = 14$, $wt(v_2G) = 15$, $wt(v_3G) = 17$, $wt(v_4G) = 14$, $\mathcal{C}_1$ is locally 2-weight (mod 3) at V_1. Next, take a 3-dimensional subspace of $\mathbb{F}_3^4$ as

$$V_3 = \{(0, a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \mathbb{F}_3\}.$$

Then, (1) holds for any $v \in V_3$ except for

$$v \in \{(0,b,0,b),(0,0,b,b),(0,0,b,2b) \mid b = 1,2\}.$$

Hence, $\mathcal{C}_1$ is 3-locally 2-weight $\pmod 3$ at V_3.

In this paper, we prove the following.

Theorem 1.4. *Let $\mathcal{C}$ be an $[n,k,d]_q$ code with $k \geq 3$, $gcd(d,q) = 1$. If $\mathcal{C}$ is e-locally 2-weight $\pmod q$ at V_m for some $1 \leq m \leq k-1$ and $e > 0$, then $e \geq q^{m-2}$.*

The structure of this paper is as follows. In Section 2, we recall the usual geometric method through projective geometry and we prove Theorem 1.4 from the geometrical point of view. In Section 3, we prove Theorem 1.2 as an application of the geometric version of Theorem 1.4 (Theorem 2.2) and Theorem 1.3.

2 Geometric preliminaries and main result

In this section, we first recall the usual geometric method [1,11,12] to investigate linear codes over $\mathbb{F}_q$ through the projective geometry. Denote by $\mathrm{PG}(r,q)$ the projective geometry of dimension r over $\mathbb{F}_q$. A *j-flat* is a projective subspace of dimension j in $\mathrm{PG}(r,q)$. The 0-flats, 1-flats, 2-flats, $(r-2)$-flats and $(r-1)$-flats are called *points, lines, planes, secundums* and *hyperplanes*, respectively. We denote by θ_j the number of points in a j-flat, i.e., $\theta_j = (q^{j+1}-1)/(q-1)$, see [13].

Let $\mathcal{C}$ be an $[n,k,d]_q$ code with generator matrix G having no all-zero column. Then, the columns of G can be considered as a multiset of n points in $\Sigma = \mathrm{PG}(k-1,q)$ denoted by $\mathcal{M}_\mathcal{C}$. A point P in Σ is called an *i-point* if it has multiplicity $m_\mathcal{C}(P) = i$ in $\mathcal{M}_\mathcal{C}$. Denote by γ_0 the maximum multiplicity of a point from Σ in $\mathcal{M}_\mathcal{C}$ and let $\mathcal{P}_i$ be the set of i-points in Σ, $0 \leq i \leq \gamma_0$. For any subset S of Σ, the *multiplicity of S with respect to $\mathcal{C}$*, denoted by $m_\mathcal{C}(S)$, is defined as

$$m_\mathcal{C}(S) = \sum_{P \in S} m_\mathcal{C}(P) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap \mathcal{P}_i|,$$

where $|T|$ denotes the number of elements in a set T. Then we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0} \mathcal{P}_i$ such that $n = m_\mathcal{C}(\Sigma)$ and

$$n - d = \max\{m_\mathcal{C}(\pi) \mid \pi \in \mathcal{F}_{k-2}\},$$

where $\mathcal{F}_j$ denotes the set of j-flats in Σ. Such a partition of Σ is called an $(n, n-d)$*-arc* of Σ. Conversely an $(n, n-d)$-arc of Σ gives an $[n, k, d]_q$ code in the natural manner. A line l with $t = m_{\mathcal{C}}(l)$ is called a *t-line*. A *t-plane*, a *t-hyperplane* and so on are defined similarly. For an m-flat Π in Σ, let

$$\gamma_j(\Pi) = \max\{m_{\mathcal{C}}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\},\ 0 \le j \le m.$$

We denote simply by γ_j instead of $\gamma_j(\Sigma)$. It holds that $\gamma_{k-2} = n-d$, $\gamma_{k-1} = n$. When $\mathcal{C}$ is Griesmer, the values $\gamma_0, \gamma_1, ..., \gamma_{k-3}$ are also uniquely determined ([14]) as follows:

$$\gamma_j = \sum_{u=0}^{j} \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \ \text{ for } 0 \le j \le k-1. \tag{2}$$

Denote by $[h_1, \ldots, h_k]$ the hyperplane of Σ defined by a non-zero vector $h = (h_1, \ldots, h_k) \in \mathbb{F}_q^k$ as $\{\mathbf{P}(p_1, \ldots, p_k) \in \Sigma \mid h_1p_1 + \cdots + h_kp_k = 0\}$, and by a_i the number of i-hyperplanes in Σ. Note that

$$a_i = A_{n-i}/(q-1) \ \text{ for } \ 0 \le i \le n-d. \tag{3}$$

The list of a_i's is called the *spectrum* of $\mathcal{C}$. We usually use τ_j's for the spectrum of a hyperplane of Σ to distinguish from the spectrum of $\mathcal{C}$. Simple counting arguments yield the following [15]:

$$\sum_{i=0}^{n-d} a_i = \theta_{k-1}, \tag{4}$$

$$\sum_{i=1}^{n-d} i a_i = n\theta_{k-2}, \tag{5}$$

$$\sum_{i=2}^{n-d} i(i-1)a_i = n(n-1)\theta_{k-3} + q^{k-2}\sum_{s=2}^{\gamma_0} s(s-1)\lambda_s. \tag{6}$$

When $\gamma_0 = 1$, one can get the following from (4)-(6):

$$\sum_{i=0}^{n-d-2} \binom{n-d-i}{2} a_i = \binom{n-d}{2}\theta_{k-1} - n(n-d-1)\theta_{k-2} + \binom{n}{2}\theta_{k-3}. \tag{7}$$

Lemma 2.1 ([16]). *Put $\epsilon = (n-d)q - n$ and $t_0 = \lfloor (w+\epsilon)/q \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Let Π be a w-hyperplane through a t-secundum δ. Then $t \le (w+\epsilon)/q$ and the following hold.*

(a) $a_w = 0$ *if an* $[w, k-1, d_0]_q$ *code with* $d_0 \geq w - t_0$ *does not exist.*

(b) $\gamma_{k-3}(\Pi) = t_0$ *if an* $[w, k-1, d_1]_q$ *code with* $d_1 \geq w - t_0 + 1$ *does not exist.*

(c) *Let* h_j *be the number of* j*-hyperplanes through* δ *other than* Π. *Then* $\sum_j h_j = q$ *and*

$$\sum_j (\gamma_{k-2} - j)h_j = w + \epsilon - qt. \tag{8}$$

From now on, let $\mathcal{C}$ be an $[n, k, d]_q$ code with $\gcd(d, q) = 1$, $q \geq 3$. We assume $k \geq 4$ since Theorem 1.4 is trivial for $k = 3$. From (3), the congruence condition in Theorem 1.3 is equivalent to that

$$m_{\mathcal{C}}(\mathit{\Pi}) \equiv n \text{ or } n - d \pmod{q} \tag{9}$$

for any hyperplane $\mathit{\Pi}$ in $\Sigma = \mathrm{PG}(k-1, q)$. Let σ be an s-flat in Σ. We call that $\mathcal{C}$ is *locally* 2*-weight* (mod q) *at* σ if (9) holds for any hyperplane $\mathit{\Pi}$ of Σ through σ. We also call that $\mathcal{C}$ is e*-locally* 2*-weight* (mod q) *at* σ if (9) holds for any hyperplane $\mathit{\Pi}$ through σ except for exactly e hyperplanes.

Example 2.1. Let $\mathcal{C}_1$ be the $[23, 4, 14]_3$ code in Example 1.1. We look at $\mathcal{C}_1$ again from the geometrical point of view. From the weight distribution, $\mathcal{C}_1$ has spectrum

$$(a_0, a_5, a_6, a_7, a_8, a_9) = (1, 1, 6, 9, 12, 11).$$

Note that $a_i > 0$ with $i \not\equiv n, n-d \pmod 3$ implies $i = 7$ and that the 7-planes are $[0,0,1,1]$, $[0,0,1,2]$, $[0,1,0,1]$, $[1,0,1,1]$, $[1,1,1,2]$, $[1,2,0,1]$, $[1,2,0,2]$, $[1,2,2,1]$, $[1,2,2,2]$, where $[a,b,c,d]$ stands for the hyperplane $V(ax_0 + bx_1 + cx_2 + dx_3)$ in $\Sigma = \mathrm{PG}(3,3)$. Let ℓ be the line through two points $\mathbf{P}(0,1,0,0)$ and $\mathbf{P}(0,0,1,0)$, and let $\delta_1 = [1,0,0,0]$, $\delta_2 = [0,0,0,1]$, $\delta_3 = [1,0,0,1]$, $\delta_4 = [1,0,0,2]$ be the planes through ℓ. Then, $m_{\mathcal{C}}(\delta_1) = m_{\mathcal{C}}(\delta_2) = 9$, $m_{\mathcal{C}}(\delta_3) = 8$, $m_{\mathcal{C}}(\delta_4) = 6$. Thus, $\mathcal{C}_1$ is locally 2-weight (mod 3) at ℓ. Next, take a point P and three 7-planes H_1, H_2, H_3 in Σ as $P = \mathbf{P}(1,0,0,0)$, $H_1 = [0,1,0,1]$, $H_2 = [0,0,1,1]$, $H_3 = [0,0,1,2]$. Then, every plane δ ($\neq H_1, H_2, H_3$) through P satisfies $m_{\mathcal{C}}(\delta) \equiv n$ or $n - d$ (mod 3). Hence, $\mathcal{C}_1$ is 3-locally 2-weight (mod 3) at P.

Theorem 2.2. *Let* σ *be an* s*-flat in* $\Sigma = \mathrm{PG}(k-1, q)$ *with* $0 \leq s \leq k-4$. *If an* $[n, k, d]_q$ *code* $\mathcal{C}$ *is* e*-locally* 2*-weight* (mod q) *at* σ *for some* $e > 0$, *then* $e \geq q^{k-s-3}$.

Proof. Let $\mathcal{C}$ be an $[n,k,d]_q$ code with a generator matrix G whose i-th row is g_i, $1 \le i \le k$. We prove Theorem 2.2 by the dual version of the usual geometric method, that is, the columns of G are considered as hyperplanes of the dual space Σ^* of $\Sigma = \mathrm{PG}(k-1,q)$. For $P = \mathbf{P}(p_1,\dots,p_k) \in \Sigma^*$, *the weight of P with respect to G*, denoted by $w_G(P)$, is defined in [17] as the weight of a corresponding codeword, i.e., $w_G(P) = wt(\sum_{i=1}^{k} p_i g_i)$. Now, let

$$F_1 = \{P \in \Sigma^* \mid w_G(P) \not\equiv 0, d \pmod q\}$$

and let σ^* be the set of hyperplanes in Σ through σ, which forms a $(k-2-s)$-flat of Σ^*. Let $\varphi_1 = |F_1 \cap \sigma^*|$. Then, it follows from "Proof of Theorem 1.5" in [18] that $\varphi_1 \ge q^{k-3-s}$. This completes the proof. □

In the above proof, the $(k-2-s)$-flat σ^* corresponds to V_m in Theorem 1.4 with $m = k-1-s$. Hence, Theorem 1.4 follows. Especially for $s = k-4$, we get the following.

Corollary 2.3. *Let $H_1, H_2, \dots, H_s$ be distinct s hyperplanes through a $(k-4)$-flat σ in Σ. Assume $m_{\mathcal{C}}(\Pi) \equiv n$ or $n+1 \pmod q$ for any other hyperplane Π containing σ. Then, $m_{\mathcal{C}}(H_i) \equiv n$ or $n+1 \pmod q$ for $1 \le s \le q-1$.*

3 Proof of Theorem 1.2

We prove the non-existence of $[g_9(4,d),4,d]_9$ codes for $d = 197, 583, 809$ as an application of Theorem 2.2.

Lemma 3.1. *There exists no $[223,4,197]_9$ code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[223,4,197]_9$ code. Then, an i-plane through a fixed t-line in $\Sigma = \mathrm{PG}(3,9)$ satisfies

$$t \le \frac{i+11}{9} \tag{10}$$

by Lemma 2.1. If there exists a 2-plane δ_2, it follows from (10) that there exists no 2-line on δ_2, a contradiction. If there exists a 11-plane, it corresponds to a $[11,3,d_0]_9$ code with $d_0 \ge 11-2 = 9$, which does not exist, see [5]. Thus, $a_{11} = 0$. In this way, one can get

$$a_i = 0 \text{ for all } i \notin \{0,1,7\text{-}10,16,17,25,26\}.$$

This procedure to rule out some possible multiplicities of hyperplanes using the known results on $n_q(k-1,d)$ and Lemma 2.1 and the possible spectra for an $(n-d)$-hyperplane is called the **first sieve** [7]. The equality (8) gives

$$\sum_j (26-j)h_j = w + 11 - 9t \tag{11}$$

with $\sum_j h_j = 9$. From (7), we have

$$\sum_{i\le 17} \binom{26-i}{2} a_i = 6705. \tag{12}$$

Suppose $a_0 > 0$ and let δ_0 be a 0-plane. Setting $w = t = 0$, the maximum possible contribution of h_j's in (11) to the LHS of (12) is $(h_{16}, h_{25}, h_{26}) = (1, 1, 7)$. Hence we get

$$(\text{LHS of (12)}) \le \binom{10}{2}\theta_2 + \binom{26}{2} = 4420,$$

a contradiction. Hence $a_0 = 0$. One can show $a_1 = 0$ similarly.

Suppose $a_9 > 0$ and let P be a 1-point on a 9-plane δ_9. Then, there are eight 2-lines and two 1-lines on δ_9 through P. Since the possible solutions of (11) with $w = 9$ is $(h_{16}, h_{25}, h_{26}) = (1, 1, 7)$ or $(h_{17}, h_{25}, h_{26}) = (1, 2, 6)$ for $t = 1$ and only $(h_{25}, h_{26}) = (2, 7)$ for $t = 2$, every plane δ $(\ne \delta_9)$ through P satisfies $m_{\mathcal{C}}(\delta) \equiv 7$ or $8 \pmod 9$, which contradicts Corollary 2.3. Thus, $a_9 = 0$. One can similarly prove $a_{10} = 0$.

Now, applying Theorem 1.3, $\mathcal{C}$ is extendable, which contradicts Theorem 1.1 (c). This completes the proof. □

Lemma 3.2 ([19])**.** *The spectrum of a* $[48, 3, 42]_9$ *code is* $(a_0, a_3, a_6) = (3, 16, 72)$.

Lemma 3.3. *There exists no* $[658, 4, 584]_9$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[658, 4, 584]_9$ code. Then, $(\gamma_0, \gamma_1, \gamma_2) = (1, 9, 74)$ from (2), and an i-plane through a fixed t-line in $\Sigma = \mathrm{PG}(3, 9)$ satisfies

$$t \le \frac{i+8}{9} \tag{13}$$

by Lemma 2.1. We have

$$a_i = 0 \text{ for all } i \notin \{0, 1, 10, 28, 37, 46\text{-}48, 55, 64, 65, 73, 74\}$$

by the first sieve. The equality (8) gives

$$\sum_j (74-j)h_j = w+8-9t \tag{14}$$

with $\sum_j h_j = 9$. Suppose $a_0 > 0$ and let δ_0 be a 0-plane. Since the possible solution of (14) with $w = t = 0$ is $(h_{73}, h_{74}) = (8,1)$ only, $\mathcal{C}$ is 1-locally 2-weight (mod 9) at any point on δ_0, which contradicts Corollary 2.3. Hence $a_0 = 0$.

Suppose $a_{48} > 0$ and let P be a 1-point on a 48-plane δ_{48}. Then, from Lemma 3.2, the lines on δ_{48} through P consist of nine 6-lines and one 3-line. Since the possible solution of (14) with $(w,t) = (48,6)$ is $(h_{73}, h_{74}) = (2,7)$ only and since the RHS of (14) with $(w,t) = (48,3)$ is 29, there are at most one 48-plane ($\neq \delta_{48}$) through P, which also contradicts Corollary 2.3. Thus, $a_{48} = 0$.

Now, we have $a_i = 0$ for all $i \not\equiv n, n-d \pmod 9$. Applying Theorem 1.3, $\mathcal{C}$ is extendable, which contradicts Theorem 1.1 (a). This completes the proof. □

Lemma 3.4 ([20])**.** *Let $\mathcal{C}$ be a Griesmer $[n,k,d]_q$ code, where q is a power of a prime p. If q divides d, then $\mathcal{C}$ is p-divisible.*

Lemma 3.5. *There exists no $[911,4,809]_9$ code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[911,4,809]_9$ code. Then, $(\gamma_0,\gamma_1,\gamma_2) = (2,12,102)$ from (2), and an i-plane through a fixed t-line in $\Sigma = \mathrm{PG}(3,9)$ satisfies

$$t \le \frac{i+7}{9} \tag{15}$$

by Lemma 2.1. Since $\gamma_0 = 2$, we have $\lambda_2 = \lambda_0 + 91$, for $\lambda_0 + \lambda_1 + \lambda_2 = \theta_3$, and $\lambda_1 + 2\lambda_2 = 911$. One can get

$$a_i = 0 \text{ for all } i \notin \{0, 47, 48, 74\text{-}81, 101, 102\}$$

by the first sieve. The equality (8) gives

$$\sum_j (102-j)h_j = w+7-9t \tag{16}$$

with $\sum_j h_j = 9$. Suppose $a_0 > 0$ and let δ_0 be a 0-plane. Then, we have $a_0 = 1$ and that the other planes are 101- or 102-plane by (16), which contradicts Corollary 2.3 taking a 0-point of δ_0 as σ. Hence $a_0 = 0$. From (7), we have

$$\sum_{i\le 81} \binom{102-i}{2} a_i = 81\lambda_2 - 4131. \tag{17}$$

Suppose $a_{81} > 0$ and let δ_{81} be an 81-plane. Then, the spectrum of δ_{81} is $(\tau_0, \tau_9) = (1, 90)$ [21]. Setting $w = 81$, the maximum possible contribution of h_j's in (16) to the LHS of (17) is $(h_{47}, h_{76}, h_{101}) = (1, 1, 7)$ for $t = 0$ and $(h_{101}, h_{102}) = (7, 2)$ for $t = 9$. Hence we get

$$\text{(LHS of (17))} \leq \binom{47}{2} + \binom{76}{2} + \binom{81}{2} = 2020,$$

whence $\lambda_2 \leq 75$, which contradicts that $\lambda_2 = 91 + \lambda_0 \geq 91$. Hence $a_{81} = 0$. One can similarly prove $a_{80} = a_{79} = 0$.

Now, let δ be a 102-plane with spectrum τ_j. Then, δ corresponds to a Griesmer $[102, 3, 90]_9$ code from (15), and hence $\tau_j > 0$ implies $j \in \{0, 3, 6, 9, 12\}$ by Lemma 3.4. Take a 1-point P on δ and let s_j be the number of j-lines through P on δ. Then, we have

$$(s_3, s_6, s_9) \in \{(1, 0, 0), (0, 1, 1), (0, 0, 3)\}. \tag{18}$$

Note that the solutions of (16) with $w = 102$ satisfy that $h_{76} + h_{77} + h_{78}$ is at most $4 - m$ for $t = 3m$ with $m = 1, 2, 3, 4$. Hence, from (18), there are at most three 76-, 77- or 78-planes through P, which contradicts Corollary 2.3. Thus, we obtain $a_{76} = a_{77} = a_{78} = 0$. Applying Theorem 1.3, $\mathcal{C}$ is extendable, which contradicts Theorem 1.1 (a). This completes the proof. $\square$

Now, Theorem 1.2 follows from Theorem 1.1 and Lemmas 3.1, 3.3, 3.5.

4 Conclusion

When an $[n + 1, k, d + 1]_q$ code with $d + 1$ divisible by q does not exist, it is most likely that an $[n, k, d]_q$ code does not exist as well. Hill-Lizak's Extension Theorem is often employed to prove the non-existence of an $[n, k, d]_q$ code with $d \equiv -1 \pmod q$. We gave a result on a new notion "e-locally 2-weight $\pmod q$" for linear codes over $\mathbb{F}_q$, which could help to rule out some possible weights of codewords so that one can apply the extension theorem to get a contradiction.

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