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# ON THE CRITICAL POINTS OF KYURKCHIEV'S METHOD FOR SOLVING ALGEBRAIC EQUATIONS

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*This paper is dedicated to Prof. Nikolay Kyurkchiev on the occasion of his 70th anniversary*

Abstract. This paper gives sufficient conditions for $k$th approximations of the zeros of polynomial $f(x)$ under which Kyurkchiev's method fails on the next step. The research is linked with an attack on the global convergence hypothesis of this commonly used in practice method (as correlate hypothesis for Weierstrass–Dochev's method). Graphical examples are presented.

**1. Introduction.** Let $f$ be a monic polynomial of degree $n$,

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

with simple roots $x_i$, $i = 1, 2, \ldots, n$.

Let $x_i^k$, $i = 1, 2, \ldots, n$, be distinct reasonably close approximations of these zeros.

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*Key words:* polynomial roots, Kyurkchiev's method, divergent sets.

Kyurkchiev [9] proposes the following method with order of convergence $\tau = 4$ as a modification of Ehrlich's method [3] with cubic order of convergence:

$$(1)\quad x_i^{k+1} = x_i^k - \frac{f(x_i^k)}{f'(x_i^k) - f(x_i^k)\sum_{j\neq i}^{n}\frac{1}{x_i^k - x_j^k} - f(x_i^k)\sum_{j\neq i}^{n}\frac{f(x_j^k)}{(x_i^k - x_j^k)^3}\prod_{s\neq i,j}^{n}\frac{1}{x_j^k - x_s^k}},$$

$$i = 1,\ldots,n;\;\; k = 0,1,2,\ldots.$$

The following theorem is valid:

**Theorem A.** *Let* $0 < q < 1$, $d = \min|x_i - x_j| > 0$ *and* $c > 0$

$$(2)\qquad c < \frac{d}{2 + en}.$$

*If the initial approximation* $\{x_i^0\}_{i=1}^n$ *satisfies the inequalities*

$$|x_i^0 - x_i| \leq cq, \quad i = 1,2,\ldots,n$$

*then the following estimate*

$$(3)\qquad |x_i^k - x_i| \leq cq^{4^k}, \quad i = 1,2,\ldots,n; \quad k = 0,1,\ldots$$

*holds.*

Following Carstensen [2] (see, also Petkovic, Herceg and Ilic [13]) we have

$$\frac{f'(x_i^k)}{f(x_i^k)} - \sum_{j\neq i}^{n}\frac{1}{x_i^k - x_j^k} = \frac{1 + \sum_{j\neq i}^{n}\frac{W_j}{x_i^k - x_j^k}}{W_i},$$

where

$$W_i = W_i(x_i^k) = \frac{f(x_i^k)}{\prod_{j\neq i}^{n}(x_i^k - x_j^k)}$$

is Weierstrass's correction.

In [14] Petkovic, Rancic and Milosevic explore all known methods with fourth order of convergence (they suggest also new methods) and give to the reader serious computer experiments.

In term of correction $W_i$, the method (1) can be written in the following way

$$x_i^{k+1} = x_i^k - \frac{W_i}{1 + \sum_{j \neq i}^{n} \frac{W_j}{x_i^k - x_j^k} + W_i \sum_{j \neq i}^{n} \frac{W_j}{(x_i^k - x_j^k)^2}}, \tag{4}$$

$$i = 1, \ldots, n; \;\; k = 0, 1, 2, \ldots .$$

In this paper we will explore one of the possible divergent sets of the method (4).

**2. Critical Points of Method (1) or (4).** Such critical initial conditions for some methods have been considered by Hristov and Kyurkchiev [4], Atanassova, Kyurkchiev and Yamamoto [1], Kanno, Kyurkchiev and Yamamoto [7], Kyurkchiev and Moskona [11], Hristov, Kyurkchiev and Iliev [5], Valchanov, Iliev and Kyurkchiev [16], Valchanov [15], Kyurkchiev [8], Iliev and Kyurkchiev [6], Kyurkchiev and Iliev [10], Petkovic and Kjurkchiev [12], Zheng and Kyurkchiev [17].

For non-attractive initial approximations that lead to divergent sets the following statement is valid:

**Suppose that for some $1 \leq i < j \leq n$, the sequence of approximations $x_1^k, \ldots, x_n^k$ satisfies the condition**

$$x_i^k - \frac{W_i}{1 + \sum_{s \neq i}^{n} \frac{W_s}{x_i^k - x_s^k} + W_i \sum_{s \neq i}^{n} \frac{W_s}{(x_i^k - x_s^k)^2}} = x_j^k - \frac{W_j}{1 + \sum_{s \neq j}^{n} \frac{W_s}{x_j^k - x_s^k} + W_j \sum_{s \neq j}^{n} \frac{W_s}{(x_j^k - x_s^k)^2}} \tag{5}$$

**then $x_i^{k+1} = x_j^{k+1}$, and thus, the $(k+2)$-th step of the (4) cannot be performed.**

**Note.** An explicit description of these divergent sets (as function of coefficients of a given polynomial and $k$-th approximations received using method

(4)) can be accomplished with the techniques proposed in the articles cited above and we will skip it.

**Numerical Examples**

1. For illustration, we consider a non-attractive set $D_f$ in the example of the equation

$$f(x) = x^2 + x + 1 = 0. \tag{6}$$

The non-attractive set, is given by

$$D_f := \frac{(x-y)^2(x^2+x+1)}{(x-y)^4 - (x-y)^2(y^2+y+1) - (x^2+x+1)(y^2+y+1)} + \\ + \frac{(x-y)^2(y^2+y+1)}{(x-y)^4 - (x-y)^2(x^2+x+1) - (x^2+x+1)(y^2+y+1)} - 1 = 0$$

(where $x = x_1^k$, $y = x_2^k$) and displayed in Fig. 1 (ContourPlot), Fig. 2 (Plot3D).

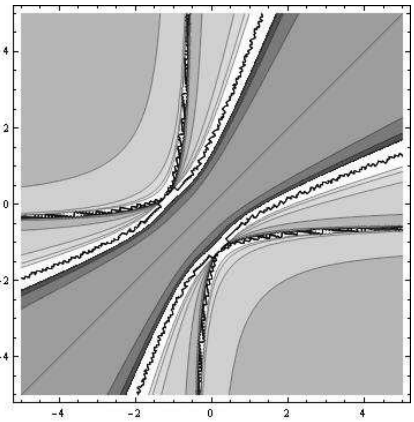


Fig. 1

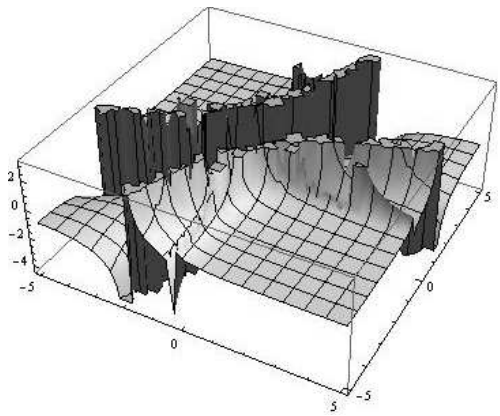

Fig. 2

2. Let us consider a quadratic equation:

$$f(x) = x^2 + ax + b = 0 \tag{7}$$

The non-attractive set $D_f$ is displayed in Fig. 3 using the program environment MATHEMATICA (function AnimatePlot3D) with parameters $a$ and $b$)

**3. Symbolic generation of a non-attractive set.** We generate non-attractive sets of a polynomial $f(x)$ of degree up to 7 using the program described below:

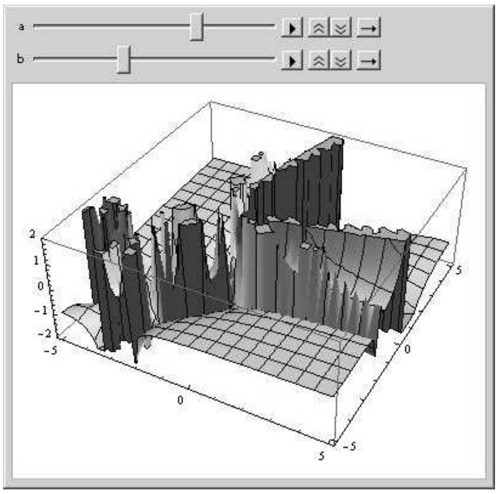


Fig. 3

```
DENOMINATOR(i, n)
    // The variable n is equal to the degree of the polynomial f(x);
    // The array W contains the expressions W_i;
    // The array v contains the names of possible
    //        variables { 'x', 'y', 'z', 's', 't', 'u', 'v' }
    exprd = '1'
    for s = 1 to n
        if s ≠ i
            exprd.ADD('+', W[s], '/', '(', v[i], '−', v[s], ')' )
    for s = 1 to n
        if s ≠ i
            exprd.ADD('+', W[i], W[s], '/', '(', v[i], '−', v[s], ')', '^', '2')
    return exprd
```

```
GENERATEEQUATIONS(func, n)
    // The string func contains the text of the polynomial f(x)
    // The array F contains the texts of the polynomials f(i), 1 ≤ i ≤ n
    for i = 1 to n
        F[i] = JOIN( '(', REPLACE(func, 'x', v[i]), ')' )
    // Generation of the expressions W_i
    for i = 1 to n
        W[i] = JOIN( '(', F[i], '/', '(' )
```

```
for j = 1 to n
    if i ≠ j
        W[i].ADD( '(', v[i], '−', v[j], ')' )
W[i].ADD( ')', ')' )
for i = 1 to n
Expr[i] = JOIN(W[i], '/', '(', DENOMINATOR(i, n), ')' )
// Generation of the equations (5)
num = 1
for i = 1 to n − 1
for j = i + 1 to n
    Equation[num] = JOIN(v[i], '−', Expr[i], '=', v[j], '−', Expr[j])
    num = num + 1
return Equation[ ]
```

The equations must be simplified in order to use the MATHEMATICA.

**4. Conclusions.** From a practical point of view the setting of the initial approximations vector is important. For any $n$th degree polynomial, this is a point in an $n$-dimensional Euclidean space. Thus, every algebraic polynomial $t$ divides the Euclidean space into two disjoint subsets $S_t$ and $R_t$. If the initial approximation belongs to $S_t$, the method is convergent, and if it belongs to $R_t$ the method is divergent. Let $S_t(r)$ and $R_t(r)$ be respectively sections of $S_t$ and $R_t$ with the sphere with center the origin and radius $r$. Obviously, the method will be globally convergent, which means that for every polynomial $t$ and for every positive $r$ the set $R_t(r)$ is a measure zero in the $n$-dimensional Euclidean space. The subset of the set $R_t(r)$ constructed in this article has measure zero, which supports the correctness of the hypothesis.

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