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EXTENDING THE CLASS OF MATHEMATICAL PROBLEMS SOLVABLE IN SCHOOL

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Abstract. The problems of practical importance which are considered in school today necessarily have to lead to a mathematical model that can be solved by school mathematics knowledge. This includes systems of equations of at most second degree, some simple trigonometry and/or some basic geometry. This restricts severely the class of such problems and conveys the impression that mathematics is not applicable enough. We provide examples of problems related to practice which are difficult to solve by means of traditional school mathematics but are amenable for solving (at least with a certain precision) with the use of software systems dealing with mathematical problems. We also present the results of an experiment with such problems that were given to school students participating in the second round of the competition "VIVA Mathematics with Computer".

1. Introduction. There is a traditionally established dividing line between the way mathematics is given in school and the manner the other subjects

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are studied. In mathematics the major tool to derive new knowledge is deduction, i.e., logical reasoning based on axioms and other already proven facts (theorems). The teaching of all other (science) subjects rests more strongly upon the "inductive" approach when the introduction of new knowledge goes through practical exploration (experimentation), analogy with something that has already been studied, extrapolation of the validity of already acquired knowledge, etc. Both approaches have their merits and, respectively, drawbacks. The deductive approach provides a solid base for absolute validity of derived facts. On the other hand, not too much of the mathematics developed in the last 250 years could be conveyed to school students with all the necessary logical rigor. The inductive approach is closer to the natural learning abilities of all living creatures and provides richer opportunities for teaching more advanced science. Both approaches are absolutely necessary, however, even within the frame of one and the same discipline. The split in education between mathematics and the other sciences was caused by the fact that, for centuries, experimentation in physics, chemistry and biology was possible through available devices, while experimentation with mathematical phenomena became widely possible only with the development of modern computers. The latter fact is still not used enough in school. This restricts strongly the opportunities to demonstrate the full power of mathematics in practical applications. By using technology and the inductive (experimental) approach in mathematics it is however possible to extend significantly the class of problems that can be considered and solved (at least with a certain precision) at school. In what follows we provide some problems with practical flavor that could be explored at earlier stages of school education by means of computer modeling and with higher mathematical rigor at later stages of the education. Such problems offer very good opportunities for demonstrating the advantages of the inquiry-based approach to teaching and learning mathematics. Several educational experiments were conducted by means of a new type of online mathematics competitions which have shown that the students are interested in (and capable of) solving such problems. The major conclusion is that the use of such problems can change positively the attitude to studying and teaching mathematics both at school and in society at large. Moreover, such problems help students develop digital competences and algorithmic thinking [4].

Here is one of the problems we have in mind.

The Problem. From a circle of radius 5 dm (Fig. 1) a box in the shape of a right rectangular prism (with a square base and without a cap) must be made by cutting off the non-colored parts. Find the maximal volume such a box can have up to the nearest hundredths of cubic dm.

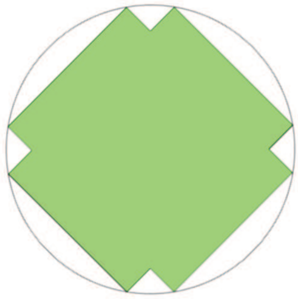

Fig. 1. The initial problem

2. A solution through a computer model based on a dynamic geometry software (GeoGebra in this case). Without loss of generality we can consider that the circle is centered at the origin O of the coordinate system. As a movable (dynamic) element in the construction we choose one of the vertices of the square (point D of Fig. 2) and consider that it moves along the vertical radius OA. The other vertices G, E and F of the square which is the base of the prism are defined as intersections of the coordinate axes with the boundary of the circle centered at O with radius OD. The altitude of the prism in this case is determined by the length of the segment DH, which is perpendicular to DG and its end H lies on the border of the original circle. The segment DH will serve as one of the vertical edges of the quadrangular prism that is to be constructed.

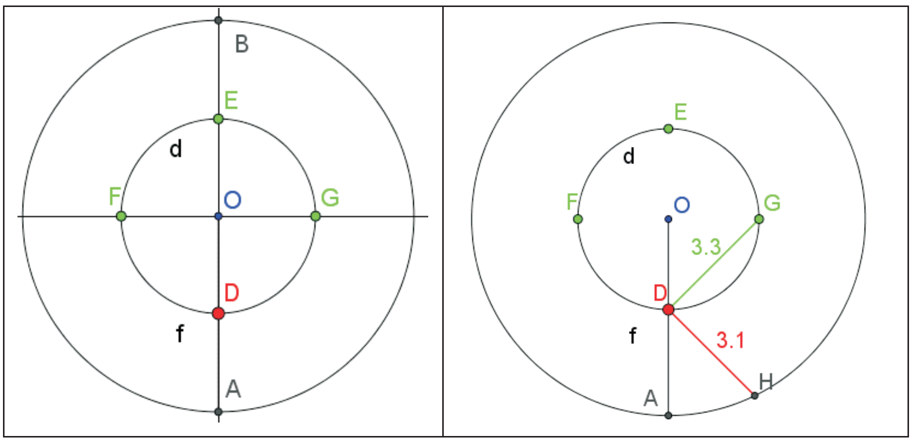

Fig. 2. Construction of the computer (dynamic) model

1 Number r	r = 5	
2 Point O	O = (0, 0)	Intersect[yAxis, xAxis]
3 Circle c	c: x² + y² = 25	Circle[O, r]
4 Point A	A = (0, -5)	Intersect[c, yAxis]
4 Point B	B = (0, 5)	Intersect[c, yAxis]
5 Segment f	f = 5	Interval[O, A]
6 Point D	D = (0, -2.62)	Point[f]
7 Circle d	d: x² + y² = 6.88	Circle[O, D]
8 Point C	C = (0, -2.62)	Intersect[d, yAxis]
8 Point E	E = (0, 2.62)	Intersect[d, yAxis]
9 Point F	F = (-2.62, 0)	Intersect[d, xAxis]
9 Point G	G = (2.62, 0)	Intersect[d, xAxis]
10 Segment g	g = 3.71	Interval[D, G]
11 Line h	h: -2.62x - 2.62y = 6....	PerpendicularLine[D, g]
12 Point H	H = (1.97, -4.6)	Intersect[c, h]
12 Point I	I = (-4.6, 1.97)	Intersect[c, h]

Fig. 3. A part of the construction protocol

All these points and segments are easily constructible by the “readymade operations” provided by the corresponding buttons of GeoGebra. A part of the construction protocol implementing this construction in GeoGebra is given in Fig. 3. The lengths of the segments DG and DH are also easy to find by means of the readymade functionalities of GeoGebra. The volume V of the prism is equal to the square of the length of DG, multiplied by the length of the DH. This quantity is automatically calculated by GeoGebra and can be shown on the screen as in Fig. 4. This is what we understand under “computer modelling of the problem”.

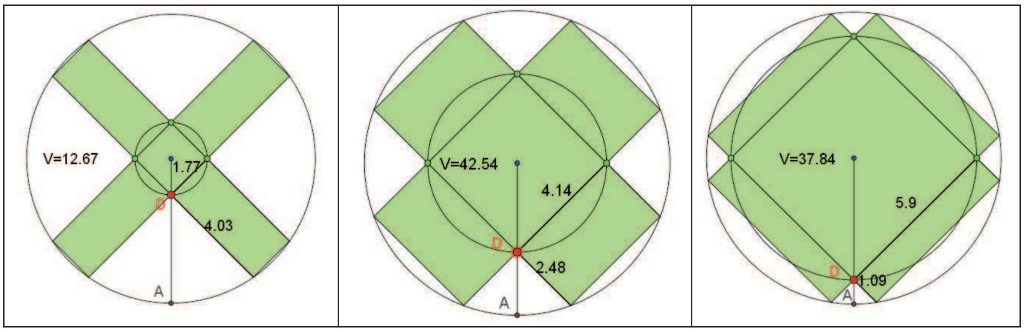

Fig. 4. Explorations with the computer model of the problem
`http://cabinet.bg/content/bg/html/d16152.html`

By moving the point D from O to A along the vertical segment OA we observe that the volume V of the prism first increases and then decreases. The maximal value we get experimentally with this dynamic computer model is 45.83 cubic dm. The size of the segment DG for this optimal volume prism is 4.9 dm and the size of the segment DH is 1.91 dm.

An obvious generalization of this problem is to use the given circle for the construction of a maximum-volume prism with a regular n-gon as a base, where n is an integer greater or equal to 3. On Fig. 5 each of the cases $n = 3$, $n = 5$ and $n = 6$ is presented by two configurations.

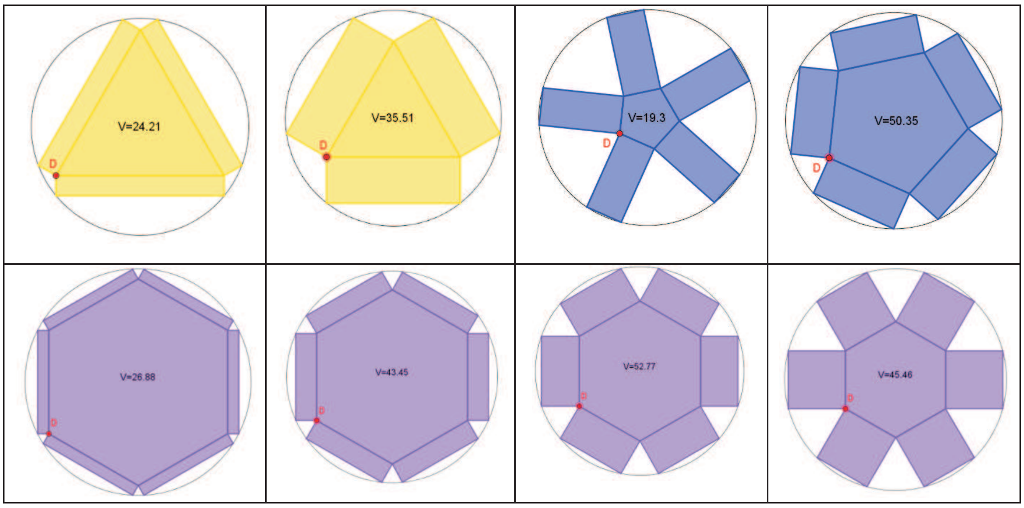

Fig. 5. Prisms with regular n-gon in the base for $n = 3$, 5, 6

To find the maximum-volume prism in each of these cases we will use again a computer model of the problem. One has to find a side, DG, of the regular n-gon, construct the altitude DH and calculate the volume of the prism as product of the area of the regular n-gon and the size of DH. Given the point D, the next vertex G of the regular n-gon could be found by rotation of D around O in counter clockwise direction to an angle equal (in radians) to 2π divided by n, or (in degrees) – to 360 divided by n. The area of the n-gon is n times the area of the triangle ODG. The area of ODG is readily available through the buttons of GeoGebra.

Experimenting with the files provided at

`http://cabinet.bg/content/bg/html/d16160.html`
`http://cabinet.bg/content/bg/html/d16165.html`
`http://cabinet.bg/content/bg/html/d16155.html`

one finds that the corresponding sizes for the maximum-volume prisms are:

- for $n = 3$ the maximal possible volume is 35.51 cubic dm, the side of the regular triangle is 6.24 dm and the altitude of the prism is 2.11;
- for $n = 5$ the maximal possible volume is 50.37 cubic dm, the side of the regular pentagon is 4.01 dm and the altitude of the prism is 1.8;
- for $n = 6$ the maximal possible volume is 52.79 cubic dm, the side of the regular hexagon is 3.39 dm and the altitude of the prism is 1.79.

A slightly better acquaintance with GeoGebra allows developing a very flexible computer model that deals simultaneously with:

- different radii r of the original circle (from which the prism is to be made);
- different number n of sides of the regular n-gon in the base of the prism;
- different radii $r_1 \leq r$ of the circle where the regular n-gon is inscribed.

First of all we observe that, given the initial radius r of the circle, the natural "dynamic object" in the construction is not the segment OD but its length which we denote by r_1. This number is the radius of the circle in which the regular n-gon (in the base of the prism) is inscribed. Any point D on the boundary of the circle of radius r_1 at center O can serve as a vertex of the regular n-gon. The construction runs as follows:

- Construct three sliders: for the radius r of the initial circle, for the radius r_1 ($r_1 \leq r$) of the circle in which the regular n-gon is inscribed, and for the integer $n > 2$.
- Construct the circle c with center at O and radius r.
- Construct the circle d with center at O and radius r_1 and select an arbitrary point D on it which will serve as a vertex of the regular n-gon.
- Construct the point G as the image of D under a counterclockwise rotation with centre at O and angle 2π divided by n (in radians) or 360 divided by n (in degrees).
- Construct the regular n-gon with side DG.
- Construct a ray with origin at D which is perpendicular to DG and goes outside d. Denote by H the intersection of this ray with the boundary of c. The segment DH will serve as a vertical edge of the prism to be constructed.
- Calculate the area of the n-gon and multiply it by the length of DH. The number obtained is the volume V of the prism corresponding to r, r_1 and n.

Given r and n one can use the slider for r_1 to find the maximal value of the volume of the prism. On Fig. 6 this is done for $r = 5$ and $n = 5$.

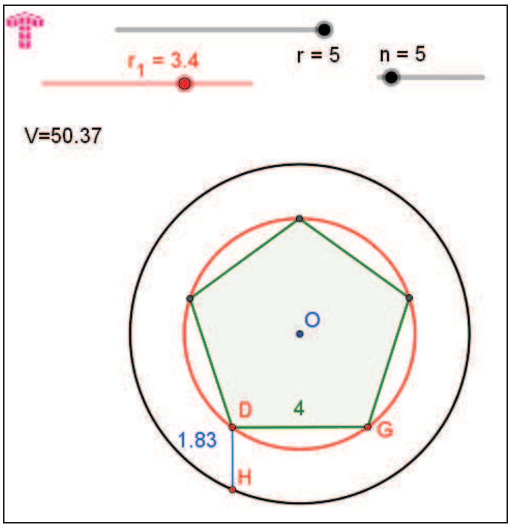

Fig. 6. Maximal volume for a pentagonal prism

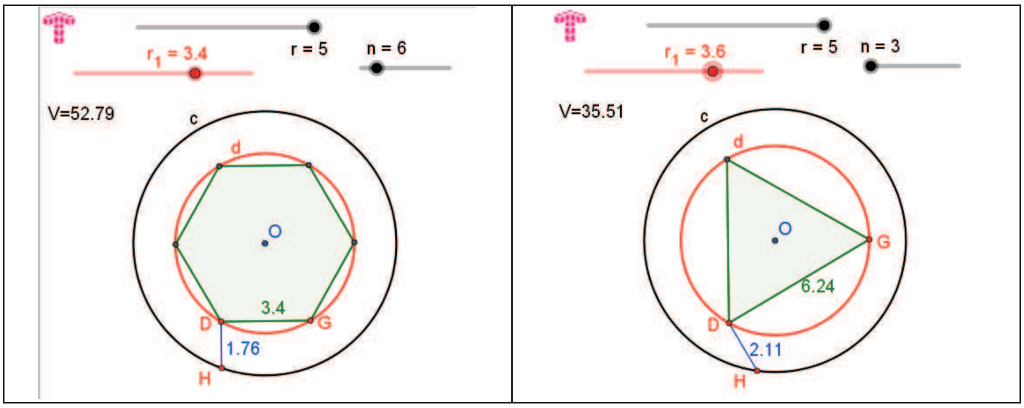

Fig. 7. Maximal volume in the case of hexagonal and triangular prisms

For the same value $r = 5$ on Fig. 7 one finds the optimal values for the volumes of a hexagonal and a triangular prism. The file producing these results is available at `http://cabinet.bg/content/bg/html/d16169.html` . The same file allows exploring other situations too. For instance, we can fix r and r_1 and see what happens with the volume V when n increases. The geometrical intuition suggests that the volume V will tend to the volume of the cylinder with radius r_1 of the base and $r - r_1$ as height (this volume is equal to $\pi r_1^2(r - r_1)$).

As we have seen, this type of "computer modelling" of the solution to the problems considered above does not require too deep mathematical knowledge. It is based on the acquaintance with the basic functionalities of GeoGebra which can be mastered to a relatively good degree by 7 – 8 graders.

We will now turn to another solution of the same group of problems which uses slightly more advanced school mathematics (Pythagoras' theorem and some elementary trigonometry).

3. A mathematical model for the maximum-volume prism. Using the already introduced notations and on the basis of Fig. 8 one gets from Pythagoras' theorem the equation

$$r^2 = (a + h)^2 + \left(\frac{b}{2}\right)^2 .$$

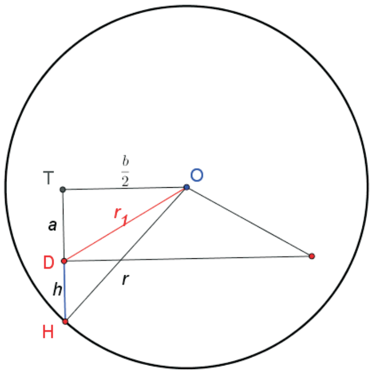

Fig. 8. Deriving the mathematical model of the problem

Here b is the size of the side DG of the regular n-gon, a is the altitude of the triangle ODG and h is the length of the segment DH. Hence,

$$h = \sqrt{r^2 - \frac{b^2}{4}} - a.$$

Expressing b and a through r_1 one gets $\frac{b}{2} = r_1 \sin \frac{\pi}{n}$; $a = r_1 \cos \frac{\pi}{n}$. Finally we derive that

$$V = \frac{n}{2} r_1^2 \sin\left(\frac{2\pi}{n}\right)\left(\sqrt{r^2 - \left(r_1 \sin\left(\frac{\pi}{n}\right)\right)^2} - r_1 \cos\left(\frac{\pi}{n}\right)\right).$$

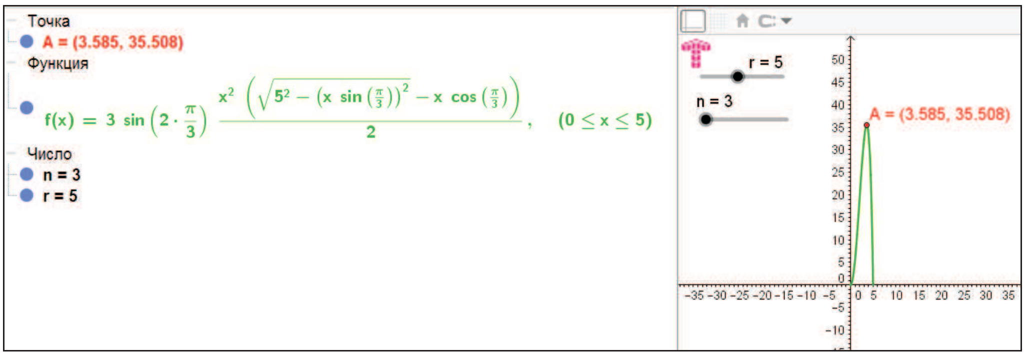

Fig. 9. The maximum-volume triangular prism

The volume V here is expressed as a function of r, r_1 and n. If r and n are given (fixed) one can draw (by means of GeoGebra) the graph of V as a function of r_1 (in the interval $[0, r]$) and see the behavior of this function. Moreover, the maximal value in this interval can be determined "automatically" by GeoGebra. For $n = 3$ and $r = 5$ we obtain (Fig. 9) that V reaches its maximum at the point with coordinates $(3.585, 35.508)$. This means that the maximal volume is 35.508 cubic dm and it is reached when the equilateral triangle in the base of the prism is inscribed in the circle of radius $r_1 = 3.585$ dm. Similarly, on Fig. 10 the cases $n = 4, 5, 6, 7$ are presented. The experimentation is conducted by means of the file at `http://cabinet.bg/content/bg/html/d16170.html`.

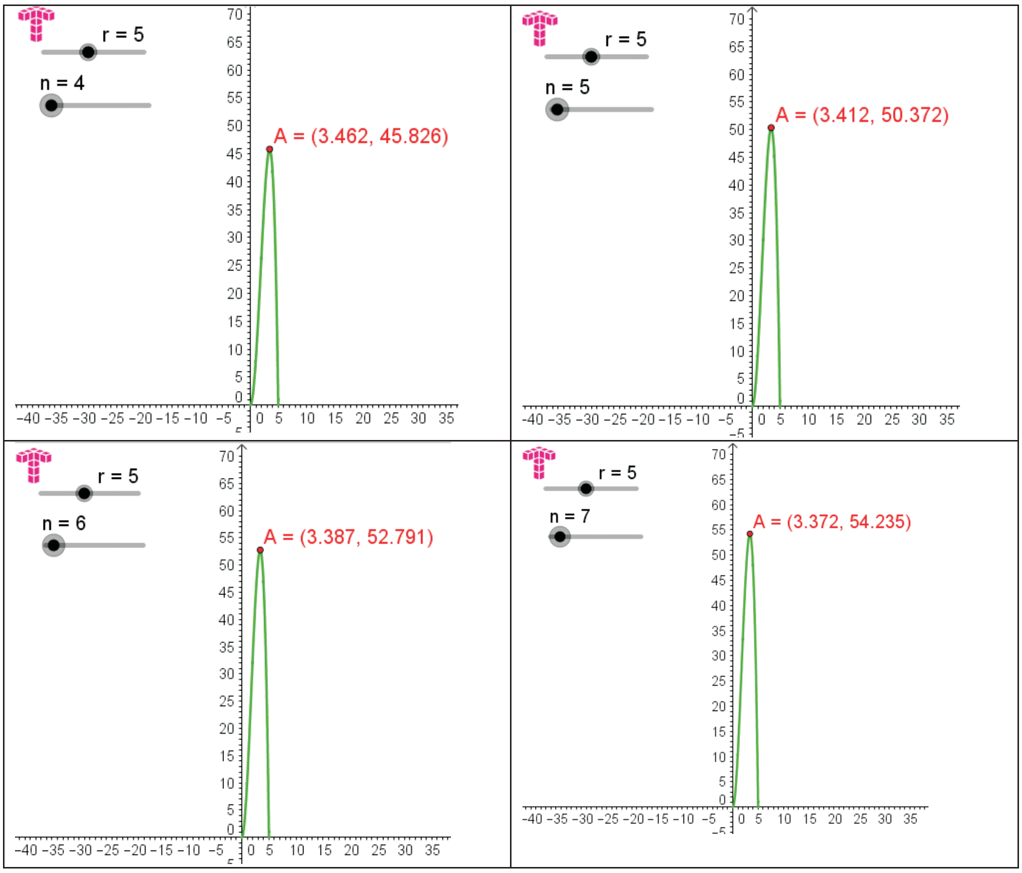

Fig. 10. Maximum-volume prisms for different n

Keeping the radius r of the original circle fixed and changing n (by moving the slide for n) one could trace the behavior of the "maximal point" A. The result of this inquiry (for $r = 1, 2, 4, 5$) is presented at Fig. 11 where the final value for n is 100 (but some red points indicate the optimal values for smaller n).

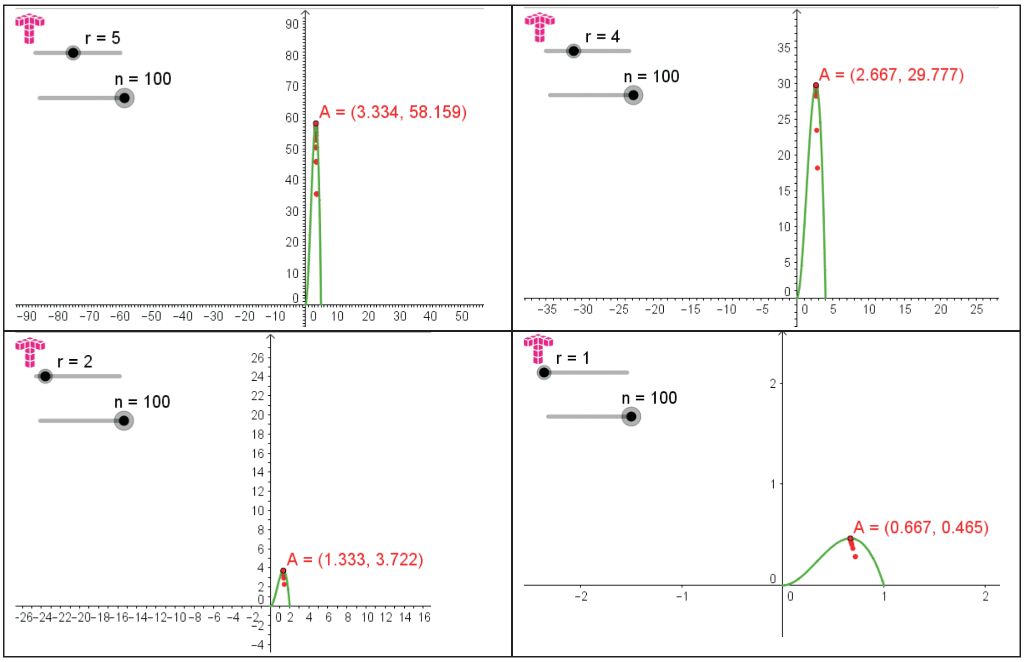

Fig. 11. Experimentation with `http://cabinet.bg/content/bg/html/d16171.html`

Another inquiry that could be made here is to put $n = 100$ and change r (by means of the slider for r). Putting the corresponding numbers in a table one gets:

r	5	4	2	1
r_1	3.334	2.667	1.333	0.667

It can be seen from this table that the ratio $\dfrac{r_1}{r}$ remains permanent:

$$\frac{r_1}{r} \approx \frac{2}{3}.$$

This is in full compliance with the geometrical intuition. For $n = 100$ the maximum-volume prism is close to a cylinder with radius r_1 of the base and height $r - r_1$. As discussed above the volume of such a cylinder is equal to $\pi r_1^2(r - r_1)$.

It is not difficult to see that the maximum of the function $\pi r_1^2(r - r_1)$ (as a function of r_1) is attained when $\dfrac{r_1}{r} = \dfrac{2}{3}$.

Still another observation can be made by students who are acquainted with the basics of calculus and know that $\dfrac{\sin x}{x}$ tends to 1 when x tends to 0. From the formula

$$V = \frac{n}{2} r_1^2 \sin\left(\frac{2\pi}{n}\right)\left(\sqrt{r^2 - \left(r_1 \sin\left(\frac{\pi}{n}\right)\right)^2} - r_1 \cos\left(\frac{\pi}{n}\right)\right)$$

one can derive that, when n goes to infinity, the volume of the optimal prism tends to $\pi r_1^2(r - r_1)$.

Because of the simple character of V as a function of r_1 (first increasing and then decreasing in the interval $[0, r]$) one can apply in the solution of our problem the derivative V' of V (with respect to r_1). It suffices to solve graphically (by means of GeoGebra) the equation $V' = 0$. Fig. 12 illustrates this solution for $r = 5$ and $n = 3$.

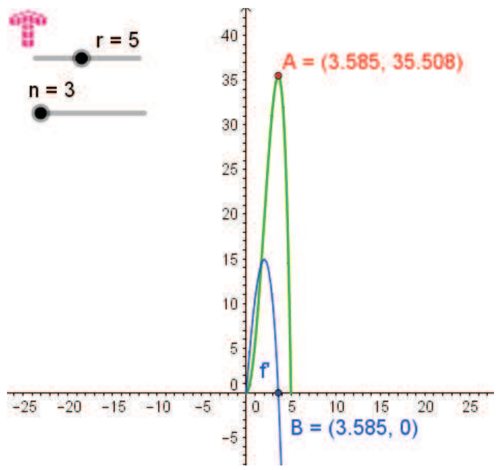

Fig. 12. A solution by means of the derivative of V
`http://cabinet.bg/content/bg/html/d16172.html`

4. The experiment with school students. In order to promote the above approaches to solving mathematical problems, difficult from the point of view of school mathematics but "tractable" by means of systems like GeoGebra, two online competitions have been implemented in Bulgaria since 2014.

These are "VIVA Mathematics with Computer" (VMC) and the "Theme of the Month" (TM). The competition VMC is for students from third to twelfth grade. The students are divided in five groups according to age (two grades in a group). It is held two times in a year (in April and December, [1]). The competition TM is for secondary shcool students. It is held every month ([3]). In VMC the participants get access for 60 minutes (through the portal `www.vivacognita.org`) to a worksheet where 10 problems are formulated. They work on them and enter the answers (usually decimal numbers) in specially provided fields in the worksheet. In TM the students get access for one month to a worksheet with 5 problems (related to one and the same mathematical theme) which are ordered in increasing difficulty. The participants are expected to explore the problems and to enter the answers (usually decimal numbers) in the respective fields of the worksheet. Some problems in both competitions are accompanied by a GeoGebra file facilitating the exploration of the mathematical essence of the problem. It is allowed to use any help. The experience gathered from the conduction of these competitions shows that solving problems similar to the above ones awakens the curiosity of students and increases their interest toward studying mathematics. Every year the best performers in these two competitions are invited to the so-called "Second Round", a "presence competition" where every student works individually without external help. In 2016 the Second Round competition was held on October 1st. It was used to run the experiment with the problems considered above. The 24 participants were divided into 4 groups according to their age (9th, 10th, 11th, 12th grade). Every student got a worksheet with 10 questions/problems that had to be answered in 90 minutes. Problem number six in each of the groups was one of the problems we considered above: in the 9th grade the problem 6 was for the rectangular prism ($n = 4$), in the 10th class the problem was for $n = 6$, in the 11th grade the problem was for $n = 3$ and in the 12th grade the problem was for $n = 5$.

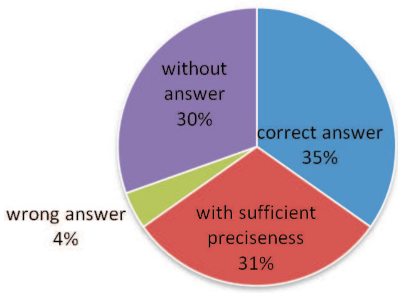

Fig. 13. Results of the Second Round of the VMC competition 2016 (Problem 6)

In the circular diagram (Fig. 13) the aggregated results (for all the 4 groups) are presented. The larger part of the participants (66%) performed well. Unexpectedly high (30%) is the group of students who did not supply any answer. Part of the participants tried to solve the problem analytically but arrived at functions of two variables and did not proceed further.

From the worksheets and the working files prepared by the students it is clear that those who solved the problem completely (or with a sufficient precision) have used computer modeling of the situation. Some of them have written down (on a paper) the intermediate results or only the maximal value of the volume (Fig. 14).

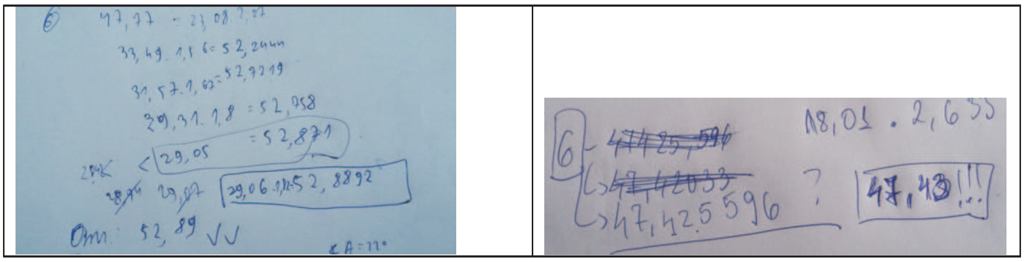

Fig. 14. Intermediate results from the computer modelling of the problem

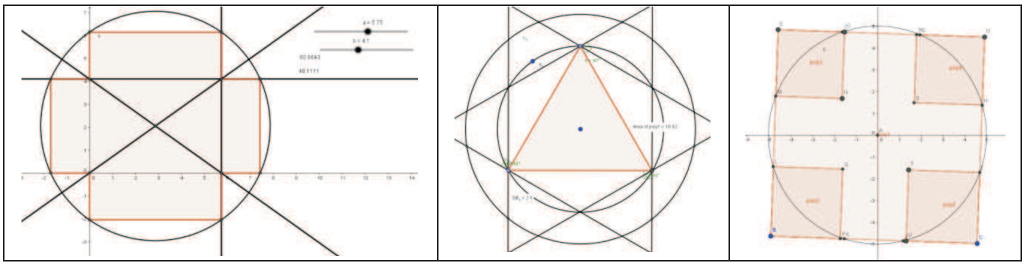

Fig. 15. Screenshots from the computer modeling of the problem by some participants

The development of the computer model with the necessary dynamic capabilities also turned to be a challenge. One of the participants used two different sliders for the two lengths of the sides of the prism base (i.e. he considered a rectangular prism base, not square). This complicated the problem. In the next three cases (Fig. 15) the dynamic properties of the computer model are not perfect but still allow finding an approximate solution (Fig. 16).

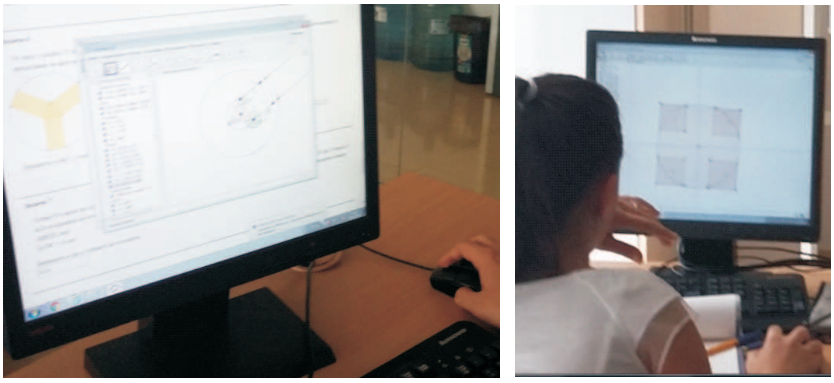

Fig. 16. During the competition

There are however participants who demonstrated good skills in the development of dynamic constructions (Fig. 17).

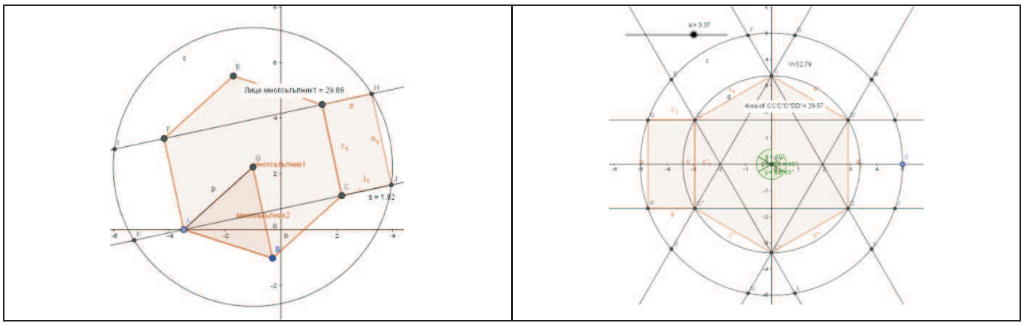

Fig. 17. Dynamic computer models of Problem 6 developed by stydents

For the computer modelling of many real-life problems one can use also the rich collection of GeoGebra files from the Virtual Mathematics Laboratory available at `http://cabinet.bg/` (see [2]).

Acknowledgments. Mathematical problems similar to the ones considered above show the importance and the advantages of using ICT in education. Prof. Ed Friedman is one of the prominent and successful promotors of this idea worldwide, including Bulgaria. We are thankful for his inspirational role in this direction and are glad to devote this article to his 80^{th} birthday.

REFERENCES

[1] Chehlarova T., P. Kenderov. Mathematics with a computer—a contest enhancing the digital and mathematical competences of the students. In: E. Kovatcheva, E. Sendova (eds.). UNESCO International Workshop: Quality of Education and Challenges in a Digitally Networked World. Za Bukvite, O'Pismeneh. Sofia, 2015, 50–62.

[2] Chehlarova T., G. Gachev, P. Kenderov, E. Sendova. A Virtual School Mathematics Laboratory. In: Proceedings of the 5th National Conference on Electronic Education, Ruse, Bulgaria, June 16–17, 2014, 146–151. `http://www.elearning-conf.eu/nkeo2014/docs/cp14/paper-21.pdf`, 11 November 2016.

[3] Kenderov P., T. Chehlarova, E. Sendova. A Web-based Mathematical Theme of the Month. *Mathematics Today*, **51** (2015), No 6, 305–309.

[4] Zehetmeier S., M. Piok, K. Holler, P. Kenderov, T. Chehlarova, E. Sendova, C. Gehring, V. Ulm. Concepts for In-Service Mathematics Teacher Education: Examples from Europe. In: C. Gehring, F. Ulm (eds). Developing Key Competences by Mathematics Education. University of Bayreuth, 2015.

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