LOWER BOUNDS ON THE DIRECTED SWEEPWIDTH OF PLANAR SHAPES[*]

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ABSTRACT. We investigate a recently introduced width measure of planar shapes called sweepwidth and prove a lower bound theorem on the sweepwidth.

1. Introduction. The computational problem SWEEPWIDTH was proposed recently [4]. The problem is to determine, given a contaminated planar shape S, a decontamination sweep that slides decontamination barriers along S, such that the maximum total length of those barriers over time is minimum. SWEEPWIDTH is a continuous two-dimensional analogue of the classical discrete graph-search problems EDGE SEARCH and NODE SEARCH (see [11], [6], and [2]). Width parameters of planar shapes have been proposed, *e. g.* elastic ringwidth [3], the major difference being that ELASTIC RINGWIDTH necessitates a single barrier. One can think of the decontamination procedure as a pursit-evasion game on S

with invisible omniscient evader that moves arbitrarily fast but cannot jump over the barriers. Many computational problems about surveillance on planar shapes are known, for example [1] and [12]. However, they are substantially different from SWEEPWIDTH.

In [4], Karaivanov *et al.* prove that to any sweep of a given width there corresponds *a canonical* sweep that slides segments, each segments having boundary endpoints; two segments never intersect except possibly at a common endpoint. Furthermore, they prove that even for a relatively simple class of orthogonal polygons called *flag polygons*, SWEEPWIDTH is **NP**-hard via reduction from PARTITION.

In the light of that discouraging result, further research focuses on the sweepwidth of specific orthogonal polygons, *e. g.* the sweepwidth of *staircase polygons* and *pyramid polygons* can be computed in linear time [13].

In this work we introduce a generalisation of SWEEPWIDTH that we call DIRECTED SWEEPWIDTH: the sweep must start on some predefined parts of the boundary and end on other predefined parts of the boundary. Using the generalisation we prove a theorem that provides a lower bound for the sweepwidth of a planar shape given three nonintersecting subshapes in it that are connected by "corridors" so that the corridor between any two of them avoids the third subshape.

2. Background. Our definitions use orthogonal polygons as fundamental planar shapes but they can be generalised to general planar shapes in an obvious way. Assume the definitions from [4] hold.

2.1. Fringed polygons and directed sweeps. Let S be an orthogonal polygon and $\mathcal{A}$ be the set of its edges. Let B_1 and B_2 be sets of straight segments such that $\forall X \in B_i \, \exists Y \in \mathcal{A} : X \subseteq Y$, for $i = 1, 2$. Furthermore, let the elements of B_1 and B_2 be pairwise disjoint except possibly for common endpoints. In this work, every segment $X_1 \in B_1$ either coincides with an edge from $\mathcal{A}$, or there is a segment $X_2 \in B_2$ such that $X_1 \cup X_2$ is a segment from $\mathcal{A}$; and likewise for the elements of B_2. We say S is *a fringed polygon with fringe* (B_1, B_2) and denote it by $S_{(B_1, B_2)}$. B_1 is called *the left fringe* and B_2, *the right fringe*. When B_1 or B_2 is a singleton, sometimes we blur the distinction between the set B_i and its sole element.

Definition 1 (corridor). *Let* $S_{(B_1, B_2)}$ *be a fringed polygon. If* $|B_1| = |B_2| = 1$ *and* B_1 *coincides with some edge of the polygon and* B_2 *coincides with another edge of the polygon, then* $S_{(B_1, B_2)}$ *is called* a corridor *and its* sides *are*

the two continuous nonintersecting curves (jagged lines) that remain after B_1 and B_2 are deleted from the boundary of S.

Note that one but not both of the sides can be a single point.

Definition 2 (fork). *Let $S_{(B_1, B_2)}$ be a fringed polygon, $|B_1| = 1$ and $|B_2| = 2$. Let each segment in the fringe coincide with a distinct edge from the boundary of S. We say that $S_{(B_1, B_2)}$ is a* fork. *Let s_1, s_2, and s_3 be the three continuous nonintersecting curves that remain after the fringe segments are deleted from the boundary of S. Let s_1 and s_2 be precisely those curves that share an endpoint with B_1. We say that s_1 and s_2 are the* sides *of the fork and s_3 is the* front *of the fork.*

Definition 3 (fringe thickness). *The fringe thickness of a corridor is the minimum distance between its two sides. If $S_{(B_1, B_2)}$ is a corridor, its fringe thickness is denoted by $fth(S_{(B_1, B_2)})$.*

A directed decontamination sweep of S with respect to (B_1, B_2), or simply a directed sweep of S when the fringe is understood, is any decontamination sweep of S that starts on B_1 and finishes on B_2. That is, the edges from B_1 are barrier points at moment 0 and the edges from B_2 are barrier points at moment 1. The bottleneck length of a directed sweep $\mathcal{Z}$ with respect to (B_1, B_2) is the supremum over time of the sum of the lenghts of all barriers during a directed sweep $\mathcal{Z}$ with respect to (B_1, B_2). The directed sweepwidth of S with respect to (B_1, B_2), shortly the directed sweepwidth of S when the fringe is understood, is the miminum over all directed sweeps with respect to that fringe. We denote it by $dsw_{(B_1, B_2)}(S)$ or simply by $dsw(S)$ when (B_1, B_2) is understood. Any of the sets in the fringe may be empty. In both are empty, it is obvious that $dsw(S) = sw(W)$. Thus, directed sweepwidth is a generalisation of sweepwidth. Consequently, given that SWEEP-WIDTH is **NP**-hard (see [4], Theorem 9), DIRECTED SWEEPWIDTH is trivially **NP**-hard as well.

[4] establishes two crucial results about decontamination sweeps. The first result, Theorem 2, says that if k is the sweepwidth of a planar shape there exists *a canonical sweep* of that shape of width k. A canonical sweep is one whose barriers are straight segments the whole time, each segment having both its endpoints on the fringe and any two segments never intersecting except possibly at a common endpoint. That result allows us to focus our attention without loss of generality on canonical sweeps only, disregarding the infinite varieties of the possible barrier shapes. Theorem 2 is immediately applicable to directed sweepwidth. So, we can

think of directed sweeps as performed by nonintersecting, except at common endpoint, straight segments with both endpoints on the fringe.

The second crucial result, though stated merely as an observation, says that sweeps are time-reversible (Observation 1). This is applicable to directed sweeps as follows: to every directed sweep with respect to the fringe (B_1, B_2) there corresponds a directed sweep with respect to the fringe (B_2, B_1) *of the same width.* In other words, the decision which one of B_1 and B_2 is the left fringe and which one, the right fringe, is purely arbitrary. Having that in mind, sometimes we define the fringe as $\{B_1, B_2\}$, thus emphasising that the two sets in the fringe are not associated with the two directions but we may associate them as we please later on.

Unfortunately, the authors of [4] were unable to prove that to every decontamination sweep there corresponds a decontamination sweep of the same width that is *progressive, i.e.,* a sweep that cleans every point precisely once. Informally speaking, progressive sweeps never go backwards. The analogous result for graph searches (see [7]) is proved by quite an involved proof and is a cornerstone in establishing the **NP**-completeness, rather than the **NP**-hardness, of the GRAPH SEARCH problem, and its equivalence to a multitude of other computational problems such as VERTEX SEPARATION (see [5]).

2.2. Linear layouts of decontamination sweeps. With respect to the flow of time, we assume that barriers keep their identity in the obvious way. Of course, new barriers may appear at some moments and existing barriers may disappear at some moments but those moments of appearance-disappearance are only a finite number, as far as the shape is a polygon. Each barrier, therefore, has *a lifetime* that is defined in the obvious way. Clearly, the lifetime of a barrier is a subinterval of $[0, 1]$. We emphasise that the barriers have identities that remains the same throughout their corresponding lifetimes.

Suppose S is a planar shape and $\mathcal{Z}$ is a decontamination sweep of it. We describe a mapping L that maps $\mathcal{Z}$ to a planar shape $L(\mathcal{Z})$ that lies in a 2D coordinate system. For every moment $t \in [0, 1]$, for every barrier b that exists in the shape at moment t, there is a vertical segment $L(b)$ of length $|b|$ in the coordinate system such that $L(b)$ is in the first quadrant at distance t from the ordinate axis. Furthermore, all those vertical segments are placed one on top of the other without gaps or overlaps, except possibly for overlaps at the endpoints. The segment at the bottom shares an endpoint with the abscissa. If one or more new barriers appear in S at moment t, their corresponding segments in the coordinate system are placed, in arbitrary order, on top of the other segments that correspond to other barriers that have existed at some previous moments. For

all barriers that have appeared before moment t, their corresponding segments are in the same relative order as they were before—that ensures continuity in $L(\mathcal{Z})$ between successive moments in time, provided that no barriers appear or disappear then. If one or more barriers disappear in S at moment t, their corresponding segments are removed from the coordinate system. There can be no gaps between the vertical segments, therefore if some segments are removed at moment t, some of the other segments may have to "move" downwards and therefore the region in the coordinate system that corresponds to the part of S that is swept by a single barrier may not be continuous.

The thus constructed planar shape $L(\mathcal{Z})$ is called *the layout of* $\mathcal{Z}$. For every $t \in [0, 1]$, the height of the layout above the abscissa equals the sum of all barriers used at moment t by $\mathcal{Z}$. Therefore, the maximum height of the layout equals precisely the bottleneck length of $\mathcal{Z}$. For additional clarity, assume that the endpoints of the vertical segments are distinguished from their interiors and so, if we consider the flow of time from 0 to 1, the endpoints of the vertical segments leave "traces" in the layout that outline the movements of the individual barriers in time.

With respect to S, $L(\mathcal{Z})$ may or may not be area preserving. It is area preserving if and only if all barriers move all the time by translation without any rotation. Furthermore, L may even not be a function between S and the
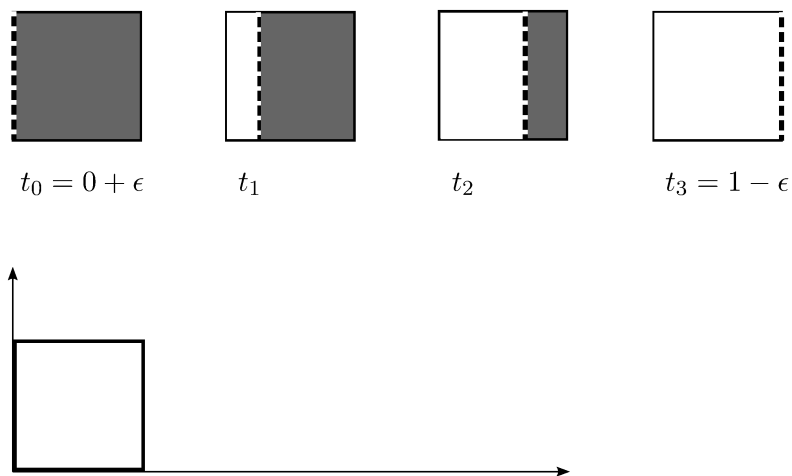

Fig. 1. A decontamination sweep for the unit square that simply slides vertical barrier without any rotation from left to right. The layout is a bijective mapping between the original square and its image. Assume ϵ is an arbitrarily small positive real
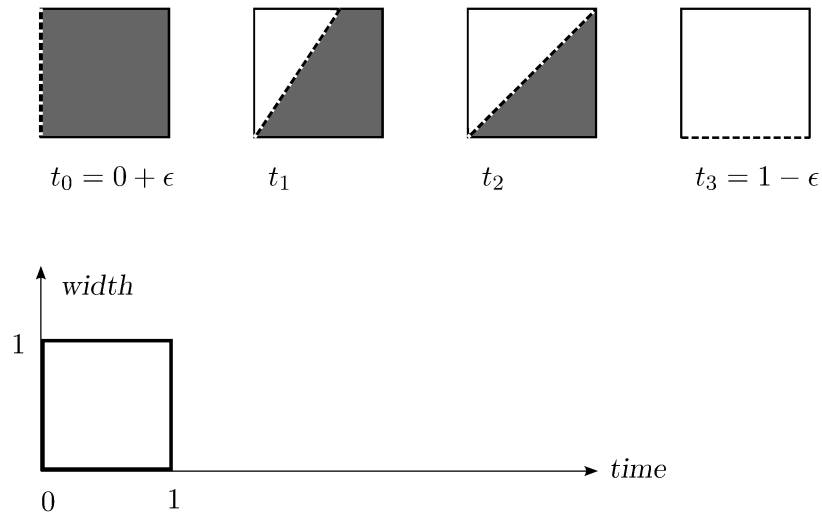
Fig. 2. A decontamination sweep for the unit square that rotates the barrier around the lower left corner. The layout maps one point, namely the lower left corner, of the original square to the whole lowest horizontal segment of the image, therefore the mapping is not even a function between the points of the two polygons

layout: if a sweeping barrier rotates around one of its endpoints during some time interval then it is mapped to a whole horizontal segment in $L(\mathcal{Z})$. For example, see Figure 2 where the lower left corner of S is mapped to the whole segment at the bottom of $L(\mathcal{Z})$.

Consider Figure 1. It shows a unit square and a decontamination sweep for it that simply slides a single unit barrier segment from left to right along it. We demonstrate four moments of the sweep: $t_0 + \epsilon$ immediately after the sweep commences, two intermediate moments t_1 and t_2, and a moment $t_3 - \epsilon$ just before the sweep ends. The contaminated part is gray and the clean part is white. The barrier is dashed. The lower paft of the figure shows the layout of the sweep. The layout is a square congruent to the original one because of the simplicity of the polygon and the sweep.

Figure 2 shows another sweep of the same square. There is a single sweeping barrier again but this time it rotates around the lower left corner. Consequently, the layout looks differently. The shapes of the two slanted sides depend on the velocity of the barrier but that is unimportant. What is important is that the maximum width of this sweep is $\sqrt{2}$ and the layout indeed attains maximum height of $\sqrt{2}$.
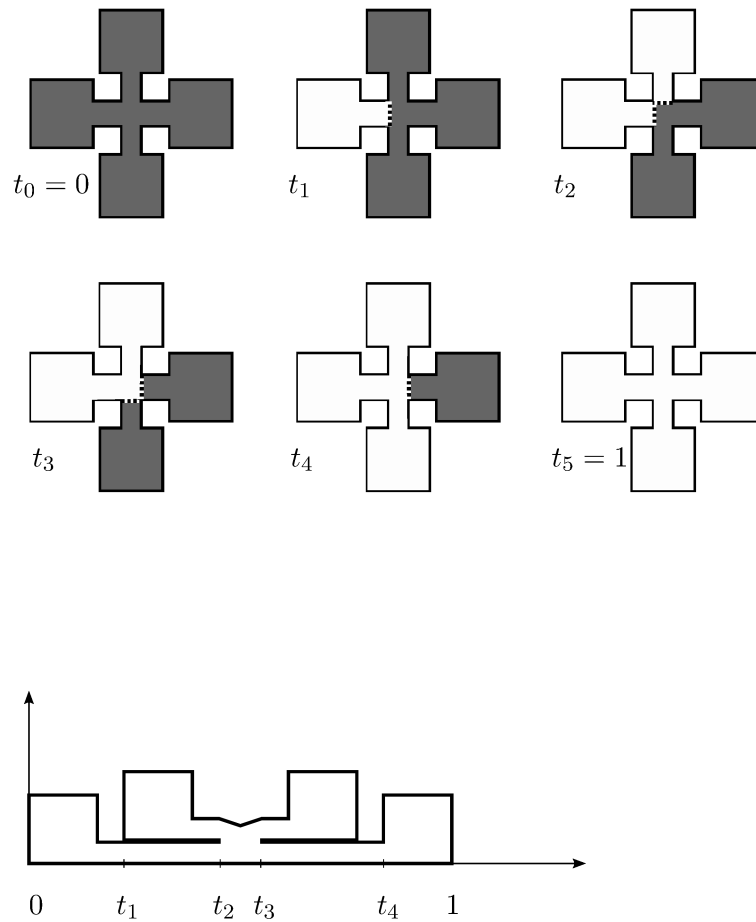
Fig. 3. A decontamination sweep of a relatively complex planar shape and the corresponding layout. The layout is drawn in blue for better clarity. Six distinct moments of the sweep are pointed out. The layout below shows all of them

A more complicated example is shown on Figure 3. The shown sweep is optimal. At some moments it uses two nonitersecting barriers. The barriers must move in a coordinated fashion in order to accomplish the decontamination. The figure shows six distinct moments of the sweep, t_0, ..., t_5, in that order in time. At $t_0 = 0$ the sweep commences. Until t_1 there is a single barrier that pushes the contamination up to "the meeting point" of the four subshapes. That barrier has to stay there for a while lest recontamination occurs in what has just been swept clean. A second barrier appears in the shape on the top, pushing the contamination out of the subshape on top until both barriers "meet" at a common endpoint at t_2. From that moment the two barriers merge and they stay merged into a single barrier until moment t_3 when the inner part of the meeting point is decontaminated. At t_3 the single barrier splits into two barriers that take care of the remaining subshapes. Note that neither the initial two subshapes (left and up), nor the final two subshapes (bottom and right) can be swept clean simultaneously because that will increase the width.

The reader may appreciate one advantage of the illustrations that use layouts. A layout depicts *the whole* decontamination process while drawing the sweep on top of the planar shape shows only a limited number of moments. On the other hand, the layout may distort the shape quite a lot, making it difficult to follow what is going on.

2.3. Layouts of directed sweeps. Suppose S is a fringed polygon with fringe (B_1, B_2). For any directed sweep $\mathcal{Z}$ on it, $L(\mathcal{Z})$ is such that $L(B_1)$ is necessarily at the left end (moment 0) and image of $L(B_2)$ is necessarily at its right end (moment 1). We say that $L(\mathcal{Z})$ is *directed layout*. Informally speaking, a directed layout is "stretched" between the images of the two fringes. Figure 4 shows a fringed polygon, a directed sweep and directed layout of its.

2.4. Subsweeps. Suppose S is a polygon and $S' \subseteq S$ is a polygon as well. Suppose $\mathcal{Z}$ is a decontamination sweep of S. Let $t_0 \in [0,1]$ be the first moment a sweeping barrier intersects S' and $t_1 \in [0,1]$ such that $t_1 > t_0$ be the last moment a sweeping barrier intersects S'. The intersection of all moving barriers of $\mathcal{Z}$ with S' over time from t_0 to t_1 is called *the subsweep of $\mathcal{Z}$ restricted to S'*. It is obvious that the said subsweep is a decontamination sweep of S' with the insignificant peculiarities that time runs from t_0 to t_1 rather than from 0 to 1. *The barriers of the subsweep at moment t*, for any $t \in [t_0, t_1]$, are the elements of the set of all barriers of $\mathcal{Z}$ at moment t with S'. Given that S' is path connected, it is clear that the set of the barriers of the subsweep at moment t is nonempty. *The bottleneck length of the subsweep of $\mathcal{Z}$ restricted to S'*, denoted by $bl(\mathcal{Z}|_{S'})$, is the supremum, over time from t_0 to t_1, of the sum of the lengths of the barriers
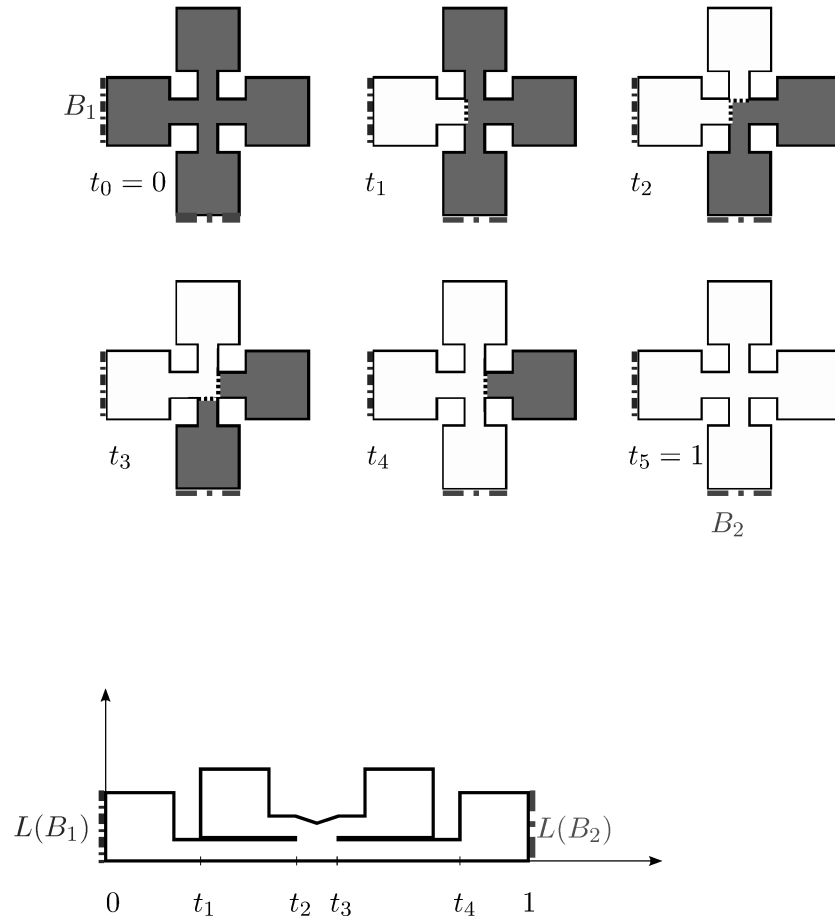
Fig. 4. A directed decontamination sweep of a fringed planar shape and the corresponding directed layout shown below. The fringe of the polygon is (B_1, B_2). B_1 and B_2 are shown is different shades of gray and different patterns of dashed lines. The layout is stretched between $L(B_1)$ and $L(B_2)$ in the obvious way. The planar shape, the sweep, and the layout are the same as the ones on Figure 3, only now the shape is fringed and the sweep and the layout is considered directed

in the subsweep.

The following result is immediately obvious.

Proposition 1. $sw(S') \leq bl(\mathcal{Z}|_{S'})$.

2.5. Sublayouts. Assume the names of Subsections 2.2, 2.3, 2.4, and 2.5 hold. *The sublayout of S' with respect to $\mathcal{Z}$ is the restriction of L to S'.* That sublayout is a layout for S', ignoring the insignificant details that its left and right extremities are at t_0 and t_1, respectively, rather than at 0 and 1, and that its bottom is not necessarily on the abscissa. A consequence of the latter is that $bl(\mathcal{Z}|_{S'})$ equals the maximum length of the intersection of the sublayout with a vertical line, rather than simply the maximum height of the sublayout.

The following definition is with respect to the names and definitions in this subsection and the former one.

Definition 4. *For any moment $t \in [t_0, t_1]$ such that the length of the intersection of the sublayout of S' with a vertical line equals $bl(\mathcal{Z}|_{S'})$, we call that intersection a bottleneck of the sublayout of S' within $L(\mathcal{Z})$.*

Figure 5 shows the same shape S we saw on Figures 3 and 4, this time with a subshape S' inside. The sweep $\mathcal{Z}$ is the same as in Figures 3 and 4. At the bottom of Figure 5 we see the sublayout of S' with respect to $\mathcal{Z}$. The subshape S' is not boundaried and so we ignore the boundaries of S.

3. A lower bound theorem. Recall the definition of "fringed polygon" and note that the same polygon can have different fringes, therefore the same ordinary (not fringed) polygon may give rise to different fringed polygons.

Theorem 1, our main result, is the continuous analogue of the following lemma from [8].

Lemma 1 ([8], Lemma 7, pp. 40)**.** *Let G be a connected graph of vertex separation $k > 1$. Let G_1, G_2, G_3 be connected, pairwise vertex-disjoint subgraphs of G, each one of them of vertex separation at least k, such that between any two G_i, G_j there is a path that is vertex-disjoint with the third one G_k. Then the vertex separation of G is at least $k + 1$.*

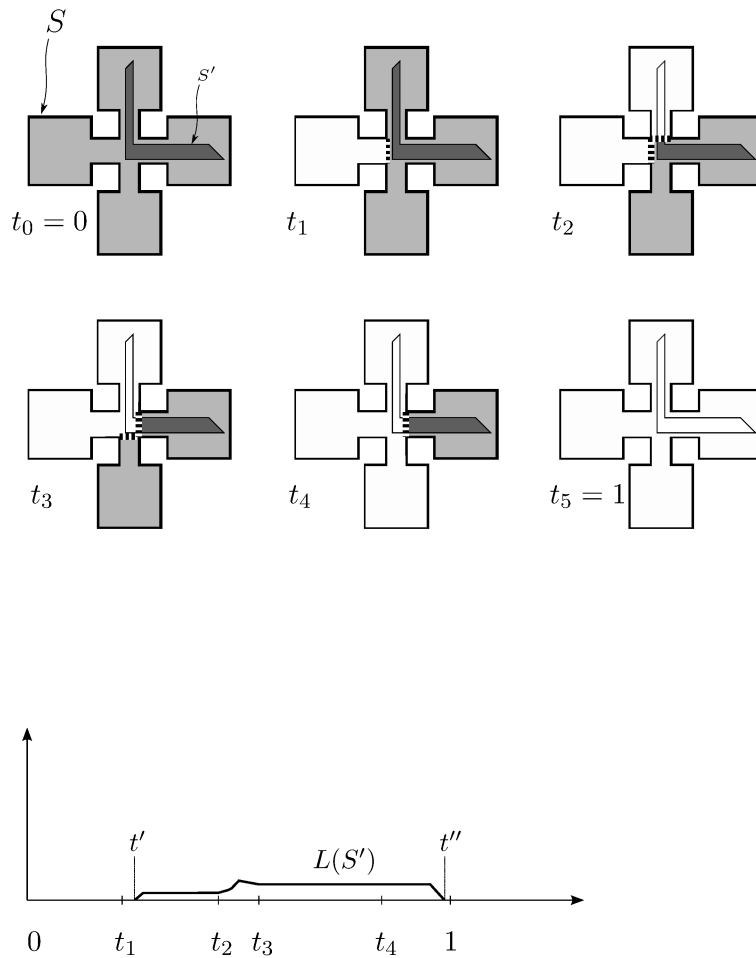**Theorem 1.** *Let S be a planar shape. Let S_1, S_2, and S_3 be three disjoint, convex polygons inside S.*

Fig. 5. The large shape is S and S' is the Γ-like subshape inside S. $\mathcal{Z}$ is the same sweep as the sweep in Figure 3 and Figure 4. At the bottom we see the sublayout $L(S')$ of S' with respect to $\mathcal{Z}$. Assume that $\mathcal{Z}$ proceeds in the most simple and obvious way between moments t_2 and t_3. The sublayout of S' starts at t' and ends at t'' where t' is the moment after t_1 when the barrier moving downwards in the upper square first touches S', and t'' is the moment before t_5 when the barrier moving rightwards in the square on the right "loses contact" with S'. For convenience, the moments $0, t_1, \ldots, t_5 = 1$ are shown on the abscissa

1. *With respect to every choice of one polygon S_i, let S_p have fringe $\{\{R_p^i\}, \emptyset\}$ for $p = 1, 2, 3$. Let there exist a fork $T_{\{\{R_i^i\}, \{R_j^i, R_k^i\}\}}^i$ inside S. Furthermore, $T^i \cap S_i = R_i^i$, $T^i \cap S_j = R_j^i$ and $T^i \cap S_k = R_k^i$.*

2. *With respect to every choice of two polygons S_i and S_j, let S_i have fringe $\{\{P_i^{i,j}\}, \emptyset\}$ and S_j have fringe $\{\{P_j^{i,j}\}, \emptyset\}$ such that a corridor $Q_{\{\{P_i^{i,j}\}, \{P_j^{i,j}\}\}}^{i,j}$ exists inside S. Furthermore, $Q^{i,j} \cap S_k = \emptyset$, $Q^{i,j} \cap S_i = P_i^{i,j}$ and $Q_{i,j} \cap S_j = P_j^{i,j}$.*

Assume that each one of the said fringes is not longer than the sweepwidth of the polygon S_i it is part of. Then the following lower bound for the sweepwidth of S holds:

$$sw(S) \geq \min\{ \max\{ sw(S_1) + fth_{\{P_2^{2,3}, P_3^{2,3}\}}(Q_{2,3}), dsw_{\{P_2^{2,3}, \emptyset\}}(S_2),$$

$$(1) \qquad\qquad dsw_{\{P_2^{2,3}, P_3^{2,3}\}}(Q_{2,3}), dsw_{\{P_3^{2,3}, \emptyset\}}(S_3)\},$$

$$\max\{ sw(S_2) + fth_{\{P_1^{1,3}, P_3^{1,3}\}}(Q_{1,3}), dsw_{\{P_1^{1,3}, \emptyset\}}(S_1),$$

$$(2) \qquad\qquad dsw_{\{P_1^{1,3}, P_3^{1,3}\}}(Q_{1,3}), dsw_{\{P_3^{1,3}, \emptyset\}}(S_3)\},$$

$$\max\{ sw(S_3) + fth_{\{P_2^{1,2}, P_1^{1,2}\}}(Q_{2,1}), dsw_{\{P_2^{1,2}, \emptyset\}}(S_2),$$

$$(3) \qquad\qquad dsw_{\{P_2^{1,2}, P_1^{1,2}\}}(Q_{2,1}), dsw_{\{P_1^{1,2}, \emptyset\}}(S_1)\},$$

$$\max\{ dsw_{\{\{R_1^1\}, \emptyset\}}(S_1),$$

$$dsw_{\{\{R_1^1\}, \{R_2^1, R_3^1\}\}}(T^1),$$

$$(4) \qquad\qquad dsw_{\{R_2^1, \emptyset\}}(S_2) + dsw_{\{R_3^1, \emptyset\}}(S_3)\},$$

$$\max\{ dsw_{\{\{R_2^2\}, \emptyset\}}(S_2),$$

$$dsw_{\{\{R_2^2\}, \{R_2^1, R_2^3\}\}}(T^2),$$

$$(5) \qquad\qquad dsw_{\{R_1^2, \emptyset\}}(S_1) + dsw_{\{R_3^2, \emptyset\}}(S_3)\},$$

$$\max\{ dsw_{\{\{R_3^3\}, \emptyset\}}(S_3),$$

$$dsw_{\{\{R_1^3\},\{R_1^3,R_1^3\}\}}(T^3),$$

(6)
$$dsw_{\{R_1^3,\emptyset\}}(S_1) + dsw_{\{R_2^3,\emptyset\}}(S_2)\},$$

(7)
$$sw(S_1) + sw(S_2) + sw(S_3)$$

$$\}$$

P r o o f. Consider any decontamination sweep $\mathcal{Z}$ of S. Consider the sub-layouts of S_1, S_2, and S_3 within $L(\mathcal{Z})$. Each of them has at least one bottleneck. Call the bottlenecks X_1, X_2, and X_3, respectively. The following cases are mutually exclusive and exhaustive.

Case 1: X_1, X_2, and X_3 happen at distinct moments, say t_1, t_2, and t_3, respectively. The possible permutations of t_1, t_2, and t_3 are 6. Having in mind the reversibility of decontamination sweeps, only 3 distinct possibilities remain. Each one of them justifies one of (1), (2), and (3). Without loss of generality, we prove only (1). It corresponds to the subcase in which $t_2 < t_1 < t_3$. Think of $L(\mathcal{Z})$. Its bottleneck length at moment t_1 is at least the length of X_1 plus the fringe thickness of the corridor $Q_{2,3}$ that connects S_2 and S_3 and avoids S_1:

(8)
$$sw(S_1) + \text{fth}_{\{P_2^{2,3},P_3^{2,3}\}}(Q_{2,3})$$

To see why that is the case, note that the said bottleneck at that moment cannot possibly be smaller that $sw(S_1)$; if it were smaller than $sw(S_1)+\text{fth}_{\{P_2^{2,3},P_3^{2,3}\}}(Q_{2,3})$ that would imply the fringe thickness of the corridor is less than $\text{fth}_{\{P_2^{2,3},P_3^{2,3}\}}(Q_{2,3})$. Of course, we keep in mind the corridor is connected so its image in the layout is connected, too.

Now think of S_2 as a fringed polygon with fringe, the single segment $P_2^{2,3}$. With respect to $L(\mathcal{Z})$, the fringe is associated with the right direction (under the assumption that $t_2 < t_3$). It follows that within the layout of S with respect to $\mathcal{Z}$:

- either the sublayout of S_2 is such that the image of $P_2^{2,3}$ left of X_1,

- or at least there is a subpolygon of S_2 that contains fully X_2 and has a single-segment fringe not shorter than $P_2^{2,3}$ that is associated with the right direction and the image of this fringe is entirely left of X_1.

It follows that the bottleneck length of $L(\mathcal{Z})$ is at least

$$(9) \qquad\qquad dsw_{\{P_2^{2,3},\emptyset\}}(S_2)$$

We prove that the bottleneck length of $L(\mathcal{Z})$ is at least

$$(10) \qquad\qquad dsw_{\{P_3^{2,3},\emptyset\}}(S_3)$$

and at least

$$(11) \qquad\qquad dsw_{\{P_2^3,P_3^2\}}(Q_{2,3})$$

in a similar fashion. Overall, (8), (9), (10), and (11) imply (1).

Case 2: Precisely two of X_1, X_2, and X_3 happen at the same moment. Assume that $t_2 = t_3 \neq t_1$. Without loss of generality, assume further that $t_1 < t_2 = t_3$. As above, we argue that:

- either the sublayout of S_1 is such that the image of R_1^1 is left of X_2 and X_3 (which are on the same vertical line),

- or at least there is a subpolygon of S_1 that contains fully X_1 and has a single-segment fringe not shorter than R_1^1 that is associated with the right direction and the image of this fringe is entirely left of at least one bottleneck of the fork T^1.

It follows that the bottleneck length of $L(\mathcal{Z})$ is at least

$$(12) \qquad\qquad dsw_{\{R_1^1,\emptyset\}}(S_1)$$

We prove that the bottleneck length of $L(\mathcal{Z})$ is at least

$$(13) \qquad\qquad dsw_{\{\{R_1^1\},\{R_2^1,R_3^1\}\}}(T^1),$$

and at least

$$(14) \qquad\qquad dsw_{\{R_2^1,\emptyset\}}(S_2) + dsw_{\{R_3^1,\emptyset\}}(S_3)\}$$

in a similar fashion. Overall, (12), (13), and (14) imply (4).

Case 3: All of X_1, X_2, and X_3 happen at the same moment. That corresponds immediately to (7). □

4. Conclusions. We have investigated a hard novel computational problem It is the continuous analogue of a well-known hard discrete problem.

It turns out the continuous problem is considerably harder—even for relatively easy results in the discrete version like Lemma 1, their analogues are difficult to formulate precisely, let alone prove rigorously.

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