Serdica J. Computing **8** (2014), No 3, 291–308

Serdica
Journal of Computing

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON THE BUSY PERIOD IN ONE FINITE QUEUE OF M/G/1 TYPE WITH INACTIVE ORBIT

Velika Dragieva

Abstract. The paper deals with a single server finite queuing system where the customers, who failed to get service, are temporarily blocked in the orbit of inactive customers. This model and its variants have many applications, especially for optimization of the corresponding models with retrials. We analyze the system in non-stationary regime and, using the discrete transformations method study, the busy period length and the number of successful calls made during it.

1. Introduction. We consider a queueing model with one server which serves N customers. Each of these customers in its free state (not being under service or blocked) produces a Poisson process of demands (calls) of the same rate λ. The customers arriving at the moments of a busy server are blocked for an exponentially distributed (with intensity μ) time interval. During this interval the customer is not allowed to make any attempts for service and is said to be blocked

ACM Computing Classification System (1998): G.3, J.7.
Key words: finite queuing systems, inactive customers, busy period, number of successful calls.

(to be in inactive state, or in the orbit of inactive customers). The service times have probability distribution function $G(x)$, with $G(0) = 0$, hazard rate function

$$\gamma(x) = \frac{G'(x)}{1 - G(x)},$$

Laplace-Stieltjes transform $g(s)$ and first moment ν^{-1}.

This model can be considered as a particular case of the models with retrials, or as a generalization of the Engset models with losses. It has many applications, both in itself and for optimization of the finite retrial queues. We may find finite queues with lost or returning customers in our daily activities, as well as in many telephone, computer and communication systems (for particular examples see [5], [2]).

The generalized Engset models have been studied in a number of papers but to the best of our knowledge they are mainly concerned with the blocking probability in the cases of multiserver system with exponential service times ([10], [9], [7]).

The steady state distributions of the system under considerations are investigated in [3]. The objective of the present paper is to investigate the busy period, which is referred to the analysis of the system at non-stationary regime. The method of analysis is similar to those in finite systems with retrials (see [1], [4], [8]).

In Section 2 we present and extend some previously obtained results concerning the distribution of the length of the busy period . Section 3 is devoted to the number of successful calls made during the busy period. A conclusion closes the paper.

2. Busy period length. Assume that the busy period starts at time $t_0 = 0$ at which there are no blocked customers and one of them generates a call. It ends at the first epoch at which the server is free and there are no blocked customers. The length of the busy period is denoted by ζ, its distribution function, $P\{\zeta \leq x\}$, by $H(x)$ and its Laplace – Stieltjes transform, by $\eta(s)$. For each $t \geq 0$ we consider the following probabilities (densities):

$$P_{1n}(t,x)dx = P\left\{\zeta > t, C(t) = 1,\, R(t) = n,\, x \leq z(t) < x + dx\right\}, \tag{2.1}$$

$$P_{1n}(t) = P\left\{\zeta > t, C(t) = 1, R(t) = n\right\}, \tag{2.2}$$

$$0 \leq n \leq N - 1,$$

$$P_{0n}(t) = P\{\zeta > t, C(t) = 0, R(t) = n\}, 1 \le n \le N-1, \tag{2.3}$$

with initial conditions

$$P_{0n}(0) = 0,\ P_{1n}(0,x) = \delta(x)\delta_{0n} \tag{2.4}$$

and Laplace transforms $\overline{P}_{in}(s)$ and $\overline{P}_{1n}(s,x)$.

Here, $C(t)$ is the number of busy servers at instant t (i.e. $C(t)$ is 0 or 1 according to whether the server is free or busy at time t), $R(t)$ is the number of inactive customers at the instant t, $z(t)$ is equal to the elapsed service time in the case of busy server, $\delta(x)$ is Dirac delta and δ_{ij} is Kronecker's delta.

Kolmogorov's equations for these transient probabilities look as follows:

$$\frac{d}{dt}P_{0n}(t) = -[(N-n)\lambda + n\mu]P_{0n}(t) + (n+1)\mu P_{0,n+1}(t) + \int_0^t P_{1n}(t,x)\gamma(x)dx,$$

$$P_{1n}(t,0) = (N-n)\lambda P_{0n}(t), \quad 1 \le n \le N-1,$$

$$\frac{\partial}{\partial t}P_{1n}(t,x) = -\left[(N-n-1)\lambda + n\mu + \gamma(x) + \frac{\partial}{\partial x}\right]P_{1n}(t,x) +$$

$$(n+1)\mu P_{1,n+1}(t,x) + (N-n)\lambda P_{1,n-1}(t,x), \quad 0 \le n \le N-1,$$

with

$$P_{0N}(t) = P_{1N}(t,x) = P_{1,-1}(t,x) = 0,$$

and initial conditions (2.4).

In addition, the following holds:

$$\frac{d}{dt}H(t) = \int_0^\infty P_{10}(t,x)\gamma(x)dx + \mu P_{01}(t),$$

$$\sum_{n=1}^{N-1} P_{0n}(t) + \sum_{n=0}^{N-1}\int_0^\infty P_{1n}(t,x)dx = 1 - H(t).$$

Applying Laplace transform in these equations , we get

$$[(N-n)\lambda + n\mu + s]\overline{P}_{0n}(s) =$$

$$(1-\delta_{n,N-1})(n+1)\mu\overline{P}_{0,n+1}(s) + \int_0^\infty \overline{P}_{1n}(s,x)\gamma(x)dx, \tag{2.5}$$

$$\overline{P}_{1n}(s,0) = (N-n)\lambda\overline{P}_{0n}(s), 1 \le n \le N-1, \tag{2.6}$$

$$\left[(N-n-1)\lambda + n\mu + \gamma(x) + s + \frac{\partial}{\partial x}\right]\overline{P}_{1n}(s,x) = \delta(x)\delta_{n0} +$$

$$\left(1-\delta_{n,N-1}\right)(n+1)\mu\overline{P}_{1,n+1}(s,x) + (N-n)\lambda\overline{P}_{1,n-1}(s,x), \tag{2.7}$$

$$0 \le n \le N-1,$$

$$\eta(s) = \int_0^\infty \overline{P}_{10}(s,x)\gamma(x)dx + \mu\overline{P}_{01}(s), \tag{2.8}$$

$$\sum_{n=1}^{N-1} \overline{P}_{0n}(s) + \sum_{n=0}^{N-1} \int_0^\infty \overline{P}_{1n}(s,x)dx = \begin{cases} \dfrac{1-\eta(s)}{s}, & \text{if } s \ne 0, \\ E[\zeta], & \text{if } s = 0. \end{cases}. \tag{2.9}$$

According to the discrete transformations method (see for example [4], [6], [8]), we rewrite equations (2.7) in a matrix form

$$[\theta I - A]\,\overline{P}_1(s,x) = D(x), \tag{2.10}$$

where θ is a scalar quantity,

$$\theta = \gamma(x) + s + \frac{\partial}{\partial x},$$

I is the identity matrix of order N, A is constructed from (2.7) in the usual way and

$$\overline{P}_1(s,x) = \left(\overline{P}_{10}(s,x), \ldots, \overline{P}_{1,N-1}(s,x)\right)^T,$$

$$D(x) = (\delta(x), 0, \ldots, 0)^T.$$

Then we transform (2.10) using the matrices Y and Λ, such that $Y^{-1}AY = \Lambda$. They are obtained in [3], where the following proposition is proved.

Proposition 1. *The matrix Λ is a diagonal matrix with elements*

$$\Lambda = \operatorname{diag}\{0, -(\mu+\lambda), \ldots, -(N-1)(\mu+\lambda)\}$$

and the entries of the k^{th} *column of* Y, $\left(y_0^{(k)}, \ldots, y_{N-1}^{(k)}\right)^T$, $k = 0, 1, \ldots, N-1$ *can be calculated by the relations*

$$y_0^{(k)} = 1, \tag{2.11}$$

$$y_n^{(k)} = \frac{-k(\lambda+\mu)}{n\mu}\left(y_0^{(k)} + \cdots + y_{n-1}^{(k)}\right) + \frac{(N-n)\lambda}{n\mu} y_{n-1}^{(k)}, \tag{2.12}$$

$$n = 1, \ldots, N-1,$$

or by their equivalent formulas

$$y_n^{(k)} = \sum_{i=0}^{n} (-1)^{n-i} \left(\frac{\lambda}{\mu}\right)^i \binom{N-k-1}{i} \binom{k}{n-i},$$

with

$$\binom{j}{l} = 0 \ \textit{if}\ l > j.$$

Furthermore, for the sum of the first n *coordinates of the* k^{th} *column we have*

$$\sum_{i=0}^{n} y_i^{(k)} = \begin{cases} \displaystyle\sum_{i=0}^{n} \left(\frac{\lambda}{\mu}\right)^i \binom{N-1}{i} \ \textit{for}\ k = 0, \\ \displaystyle\sum_{i=0}^{n} (-1)^{n-i} \left(\frac{\lambda}{\mu}\right)^i \binom{N-k-1}{i} \binom{k-1}{n-i}, \\ \textit{for}\ k = 1, \ldots, N-1 \end{cases}$$

and therefore

$$\sum_{i=0}^{N-1} y_i^{(k)} = \begin{cases} \left(1 + \dfrac{\lambda}{\mu}\right)^{N-1} \ \textit{for}\ k = 0 \\ 0 \ \textit{for}\ k = 1, \ldots, N-1. \end{cases} \tag{2.13}$$

Thus, applying in (2.10) the transformations

$$\overline{P}_1(s,x) = Y \overline{Q}_1(s,x) \tag{2.14}$$

we get it in the simpler form

$$[\theta I - \Lambda]\, \overline{Q}_1(s,x) = Y^{-1} D. \tag{2.15}$$

Because only the first coordinate of the vector D is nonzero it is sufficient to find only the first column of the matrix Y^{-1}, $\left(\overline{y}_0^{(0)}, \ldots, \overline{y}_{N-1}^{(0)}\right)^T$, for which we have:

$$\overline{y}_k^{(0)} = \binom{N-1}{k} \left(\frac{\lambda}{\mu}\right)^k \left(\frac{\mu}{\lambda+\mu}\right)^{N-1}. \tag{2.16}$$

Equation (2.15) and relation (2.14) allow to express the functions $\overline{P}_{1n}(s,x)$ in terms of N unknown quantities, the initial values $\overline{Q}_{1n}(s,0)$. Then, from (2.6) and (2.8) we express $\overline{P}_{0n}(s)$ and $\eta(s)$ in terms of the same unknowns, $\overline{Q}_{1n}(s,0)$ and with the help of (2.5) and (2.9) derive a system of linear equations for $\overline{Q}_{1n}(s,0)$. Thus, the following theorem holds.

Theorem 1. *The Laplace transforms $\overline{P}_{1n}(s,x), \overline{P}_{in}(s)$ of the probabilities $P_{1n}(t,x)$, $P_{in}(t)$, $i=0,1$ and the Laplace–Stieltjes transform, $\eta(s)$, of the busy period distribution function can be calculated by the formulas*

$$\overline{P}_{1n}(s,x) = [1-G(x)] \sum_{k=0}^{N-1} y_n^{(k)} e^{-[k(\lambda+\mu)+s]x} \left[\overline{Q}_{1k}(s,0) + \overline{y}_k^{(0)}\right], \tag{2.17}$$

$$\overline{P}_{1n}(s) = \int_0^\infty \overline{P}_{1n}(s,x)dx =$$

$$\sum_{k=0}^{N-1} y_n^{(k)} \frac{1-g_k(s)}{k(\lambda+\mu)+s} \left[\overline{Q}_{1k}(s,0) + \overline{y}_k^{(0)}\right], 0 \le n \le N-1, \tag{2.18}$$

$$\overline{P}_{0n}(s) = \frac{1}{(N-n)\lambda} \sum_{k=0}^{N-1} y_n^{(k)} \overline{Q}_{1k}(s,0), 1 \le n \le N-1. \tag{2.19}$$

$$\eta(s) = \sum_{k=0}^{N-1} \overline{Q}_{1k}(s,0) \left[1 + g_k(s) - \frac{k(\lambda+\mu)}{(N-1)\lambda}\right] + \sum_{k=0}^{N-1} g_k(s) \overline{y}_k^{(0)}, \tag{2.20}$$

where the initial conditions $\overline{Q}_{1k}(s,0)$ satisfy the following system of linear equations

$$\sum_{k=0}^{N-1} \overline{Q}_{1k}(s,0) \left\{ y_n^{(k)} \left[\delta_{n,N-1} + \frac{n\mu+s}{(N-n)\lambda} - g_k(s)\right] + \right.$$

$$(1-\delta_{n,N-1})\,\frac{k(\lambda+\mu)}{(N-n-1)\lambda}\left(y_0^{(k)}+\cdots+y_n^{(k)}\right)\Bigg\} = \sum_{k=0}^{N-1} g_k(s)y_n^{(k)}\overline{y}_k^{(0)}, \tag{2.21}$$

$$1 \le n \le N-1,$$

$$\overline{Q}_{10}(s,0)\left\{1+g_0(s)+\sum_{n=1}^{N-1}\frac{sy_n^{(0)}}{(N-n)\lambda}+[1-g_0(s)]\left(\frac{\lambda+\mu}{\mu}\right)^{N-1}\right\}+$$

$$\sum_{k=1}^{N-1}\overline{Q}_{1k}(s,0)\left\{1+g_k(s)+\sum_{n=1}^{N-1}\frac{sy_n^{(k)}}{(N-n)\lambda}-\frac{k(\lambda+\mu)}{(N-1)\lambda}\right\}=$$

$$g_0(s)-\sum_{k=0}^{N-1}g_k(s)\overline{y}_k^{(0)}. \tag{2.22}$$

Here $y_n^{(k)}$ *and* $\overline{y}_n^{(0)}$ *are given by* (2.11), (2.12) *and* (2.16), *respectively,* $g_k(s) = g(k(\lambda+\mu)+s)$.

P r o o f. The kth of the equations (2.15) has the form

$$\frac{\partial}{\partial x}\overline{Q}_{1k}(s,x)+[k(\lambda+\mu)+\gamma(x)+s]\,\overline{Q}_{1k}(s,x)=\delta(x)\overline{y}_k^{(0)},$$

$$0 \le k \le N-1,$$

with solutions

$$\overline{Q}_{1k}(s,x)=[1-G(x)]e^{-[k(\lambda+\mu)+s]x}\left[\overline{Q}_{1k}(s,0)+\overline{y}_k^{(0)}\right].$$

Thus, substituting in (2.14) and (2.6) we obtain formulas (2.17)–(2.19) for the quantities $\overline{P}_{1n}(s,x), \overline{P}_{1n}(s)$ and $\overline{P}_{0n}(s)$. Further, we substitute with these expressions in relations (2.5),

$$\sum_{k=0}^{N-1}\overline{Q}_{1k}(s,0)\left\{y_n^{(k)}\left[1+\frac{n\mu+s}{(N-n)\lambda}-g_k\right]-\right.$$

$$(1-\delta_{n,N-1})\,\frac{(n+1)\mu}{(N-n-1)\lambda}y_{n+1}^{(k)}\Bigg\} = \sum_{k=0}^{N-1} g_k y_n^{(k)}\overline{y}_k^{(0)}, \tag{2.23}$$

$$1 \le n \le N-1,$$

then substitute $y_{n+1}^{(k)}$ according to (2.12) and obtain equations (2.21). By analogy, substituting in (2.8) according to (2.17)–(2.19),

$$\eta(s) = \sum_{k=0}^{N-1} \overline{Q}_{1k}(s,0) \left[y_0^{(k)} g_k(s) + \frac{\mu}{(N-1)\lambda} y_1^{(k)} \right] +$$

$$\sum_{k=0}^{N-1} y_0^{(k)} g_k(s) \overline{y}_k^{(0)}$$

and then $y_0^{(k)}$ and $y_1^{(k)}$ according to (2.12),

$$y_0^{(k)} = 1,$$

$$y_1^{(k)} = \frac{-k(\lambda+\mu)}{\mu} + \frac{(N-1)\lambda}{\mu}$$

we get (2.20). At the end, to verify (2.22) we substitute in the normalizing condition (2.9) with (2.17)–(2.19), (2.20) and for $s \neq 0$ obtain

$$\sum_{k=0}^{N-1} \overline{Q}_{1k}(s,0) \left\{ 1 + g_k(s) - \frac{k(\lambda+\mu)}{(N-1)\lambda} + s y_0^{(k)} \frac{1-g_k(s)}{k(\lambda+\mu)+s} + \right.$$

$$\left. s \sum_{n=1}^{N-1} \frac{y_n^{(k)}}{(N-n)\lambda} + \frac{1-g_k(s)}{k(\lambda+\mu)+s} s \sum_{n=1}^{N-1} y_n^{(k)} \right\} = 1 -$$

$$s \sum_{k=0}^{N-1} \frac{1-g_k(s)}{k(\lambda+\mu)+s} \overline{y}_k^{(0)} \sum_{n=0}^{N-1} y_n^{(k)} - \sum_{k=0}^{N-1} g_k(s) \overline{y}_k^{(0)}.$$

For the sums $\sum_{n=1}^{N-1} y_n^{(k)}$ we apply relations (2.13) and the last equation gives (2.22). For $s = 0$ equation (2.22) follows from (2.8). □

Thus, to calculate $\eta(s)$ we have to find the solutions $\overline{Q}_{1k}(s,0)$ of the linear system (2.21)–(2.22). Further, upon suitable differentiations in (2.20)–(2.22) we can obtain formulas for computing the first moments of the busy period length, ζ.

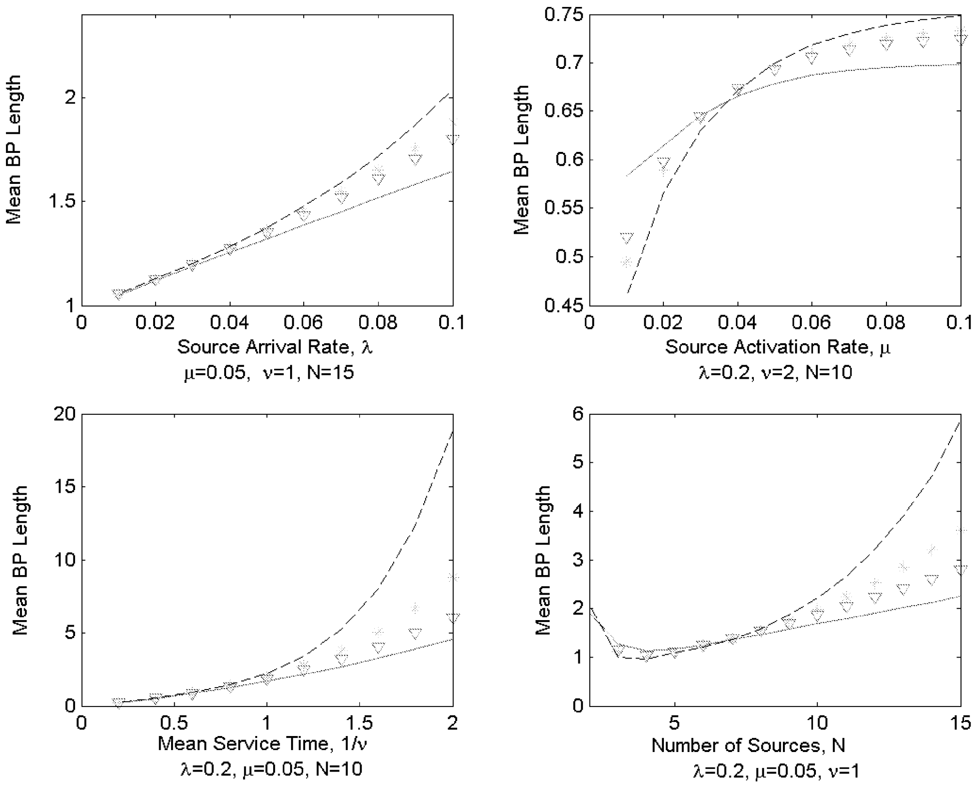

Fig. 1. Mean busy period length vs. system parameters

Besides this way, the mean busy period can be calculated with the help of formula (2.9),

$$E[\zeta] = \sum_{n=1}^{N-1} \overline{P}_{0n}(0) + \sum_{n=0}^{N-1} \overline{P}_{1n}(0). \tag{2.24}$$

If substitute here $\overline{P}_{0n}(0)$ and $\overline{P}_{1n}(0)$ according to (2.18)–(2.19), we get

$$E[\zeta] = \sum_{k=0}^{N-1} \overline{Q}_{1k}(0,0) \left\{ \sum_{n=1}^{N-1} \frac{y_n^{(k)}}{(N-n)\lambda} + \frac{\delta_{k0}}{\nu} \left(\frac{\lambda+\mu}{\mu} \right)^{N-1} \right\} + \frac{1}{\nu}, \tag{2.25}$$

with $\overline{Q}_{1k}(0,0)$-solutions of (2.21)–(2.22) for $s = 0$.

In Figure 1 we see the behaviour of the mean busy period length, $E[\zeta]$ against each one of the system parameters:

- Source arrival rate, λ (the upper-left corner);
- Source activation rate, μ (the upper-right corner);
- Mean service time, $1/\nu$ (the lower-left corner);
- Number of customers, N (the lower-right corner).

The presented results are calculated for different distributions of the service time with the same mean, $1/\nu$:

- Deterministic distribution, equal to $1/\nu$, presented with dashed lines;
- Erlang distribution with parameter 4 and 4ν, presented with lines of stars;
- Exponential distribution with parameter ν, presented with solid lines;
- Uniform distribution in the interval $(0, 2/\nu)$, presented with lines of triangles.

The graphs show that the mean busy period length is an increasing function of the source arrival rate λ, source activation rate, μ, mean service time $1/\nu$, while as a function of the number of sources, N it has a point of local minimum. We can also see that the increase of each parameter increases the influence of the type of the service distribution.

3. Analysis of the successful calls made during the busy period. Suppose that at moment $t = 0$ a busy period starts, i.e., that all customers are in free state and one of them generates a call. We denote the number of successful calls made during the time interval $(0, t)$, $t \geq 0$ by $N^S(t)$, the number of successful calls made during the busy period by N^{SBP}, the length of the busy period during which k successful calls occur by ζ_k, and consider the probabilities (densities)

$$P^S_{1nk}(t,x)dx = P\left\{\zeta > t, N^S(t) = k, C(t) = 1,\ R(t) = n, x \leq z(t) < x + dx\right\},$$

$$P^S_{ink}(t) = P\left\{\zeta > t, N^S(t) = k, C(t) = i, R(t) = n\right\}, \quad i = 0, 1,$$

$$P^{SBP}_k = P\left\{N^{SBP} = k\right\},$$

$$h_k(t) = \frac{dP\left\{\zeta_k \leq t\right\}}{dt} = \frac{dP\left\{\zeta \leq t, N^{SBP} = k\right\}}{dt}$$

with Laplace transforms $\overline{P}^S_{1nk}(s,x)$, $\overline{P}^S_{0nk}(s)$, $\overline{P}^S_{1nk}(s)$ and $\overline{h}_k(s)$.Then for the distribution of the successful calls, made during the busy period we have

$$P_k^{SBP} = P\left\{N^{SBP} = k\right\} = \int\limits_0^\infty h_k(t)dt = \overline{h}_k(0).$$

To calculate $\overline{h}_k(0)$ we first derive Kolmogorov's equations for the probabilities (densities) $P^S_{1nk}(t,x)$ and $P^S_{0nk}(t)$,

$$\frac{d}{dt}P^S_{0nk}(t) = -[(N-n)\lambda + n\mu]P^S_{0nk}(t) + (n+1)\mu P^S_{0,n+1,k}(t) + \int_0^t P^S_{1nk}(t,x)\gamma(x)dx,$$

$$P^S_{1nk}(t,0) = (1-\delta_{k1})(N-n)\lambda P^S_{0,n,k-1}(t), k \geq 1, 1 \leq n \leq N-1,$$

$$\frac{\partial}{\partial t}P^S_{1nk}(t,x) = -\left[(N-n-1)\lambda + n\mu + \gamma(x) + \frac{\partial}{\partial x}\right] P^S_{1nk}(t,x) +$$

$$(n+1)\mu P^S_{1,n+1,k}(t,x) + (N-n)\lambda P^S_{1,n-1,k}(t,x), 0 \leq n \leq N-1,$$

with

$$P^S_{0Nk}(t) = P^S_{1Nk}(t,x) = P^S_{1,-1,k}(t,x) = 0,$$

and

$$P^S_{0nk}(0) = 0, P^S_{1nk}(0,x) = \delta(x)\delta_{(n,k),(0,1)}.$$

Besides these equations the following relations hold

$$h_k(t) = \int\limits_0^\infty P^S_{10k}(t,x)\gamma(x)dx + \mu P^S_{01k}(t),$$

$$\sum_{n=0}^{N-1} P^S_{0nk}(t) + \sum_{n=0}^{N-1} P^S_{1nk}(t) = \sum_{q=k}^{\infty} \int\limits_t^\infty h_q(x)dx.$$

Applying Laplace transforms we get

$$[(N-n)\lambda + n\mu + s]\overline{P}^S_{0nk}(s) =$$

$$(3.1)\qquad \left(1-\delta_{n,N-1}\right)(n+1)\mu\overline{P}^{S}_{0,n+1,k}(s)+\int_0^\infty \overline{P}^{S}_{1nk}(s,x)\gamma(x)dx,$$

$$(3.2)\qquad \overline{P}^{S}_{1nk}(s,0)=(1-\delta_{k1})(N-n)\lambda\overline{P}^{S}_{0,n,k-1}(s),$$

$$1\le n\le N-1,$$

$$[(N-n-1)\lambda+n\mu+\gamma(x)+s+\frac{\partial}{\partial x}]\overline{P}^{S}_{1nk}(s,x)=\delta(x)\delta_{(n,k)(0,1)}+$$

$$(3.3)\qquad \left(1-\delta_{n,N-1}\right)(n+1)\mu\overline{P}^{S}_{1,n+1,k}(s,x)+(N-n)\lambda\overline{P}^{S}_{1,n-1,k}(s,x),$$

$$0\le n\le N-1, k\ge 1,$$

$$(3.4)\qquad \overline{h}_k(s)=\int\limits_0^\infty \overline{P}^{S}_{10k}(s,x)\gamma(x)dx+\mu\overline{P}^{S}_{01k}(s),$$

$$\sum_{n=0}^{N-1}\overline{P}^{S}_{0nk}(s)+\sum_{n=0}^{N-1}\overline{P}^{S}_{1nk}(s)=\frac{1}{s}\left\{\sum_{q=k}^{\infty}\left[\overline{h}_q(0)-\overline{h}_q(s)\right]\right\}=$$

$$(3.5)\qquad \frac{1}{s}\left\{1-\eta(s)-(1-\delta_{k1})\sum_{q=1}^{k-1}\left[\overline{h}_q(0)-\overline{h}_q(s)\right]\right\},\ \text{if } s\neq 0$$

and

$$(3.6)\qquad \sum_{n=0}^{N-1}\overline{P}^{S}_{0nk}(s)+\sum_{n=0}^{N-1}\overline{P}^{S}_{1nk}(s)=\sum_{q=k}^{\infty}E[\zeta_q],\ \text{if } s=0.$$

The system (3.1)–(3.6) is similar to the system (2.5)–(2.9), determining Laplace transform of the busy period and its solutions can be found with the help of the same discrete transformation, presented in Proposition 1. Solving this system successively for $k=1,2,\dots$ we can calculate each probability of the distribution of the successful calls, made during the busy period.

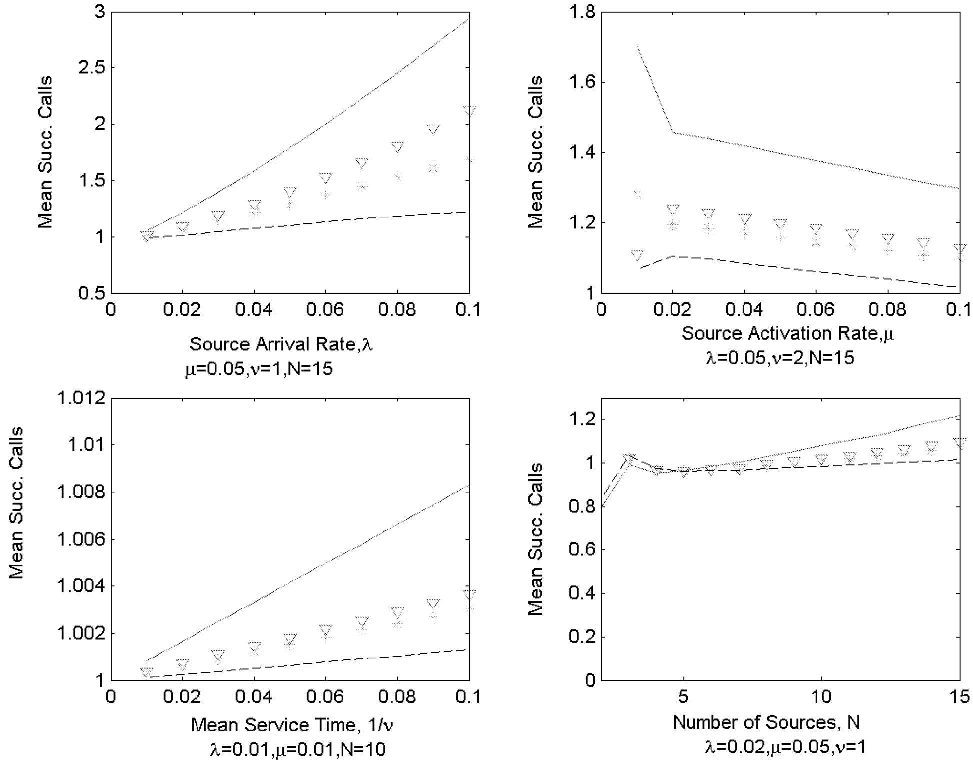

Fig. 2. Mean number of successful calls vs. system parameters

Now, we turn our attention to the computation of the mean value of this distribution. Define

$$m_{in}(t) = \sum_{k=1}^{\infty} k P_{ink}^{S}(t),$$

$$m(t) = \sum_{k=1}^{\infty} k h_k(t)$$

with Laplace transforms $\overline{m}_{in}(s)$, $\overline{m}(s)$ and

$$E\left[N^{SBP}\right] = \sum_{k=1}^{\infty} k P\left\{N^{SBP} = k\right\} = \sum_{k=1}^{\infty} k \overline{h}_k(0) = \overline{m}(0).$$

To derive equations for $\overline{m}_{in}(s)$, $\overline{m}(s)$ we multiply each of the equations (3.1)–(3.4) by k and sum over $k = 1, 2, \ldots$:

$$[(N-n)\lambda + n\mu + s]\overline{m}_{0n}(s) =$$

$$(1-\delta_{n,N-1})\,(n+1)\mu\overline{m}_{0,n+1}(s) + \int_0^\infty \overline{m}_{1n}(s,x)\gamma(x)dx, \tag{3.7}$$

$$\overline{m}_{1n}(s,0) = (N-n)\lambda\overline{m}_{0n}(s) + \overline{P}_{0n}(s), \tag{3.8}$$

$$1 \le n \le N-1,$$

$$[(N-n-1)\lambda + n\mu + \gamma(x) + s + \frac{\partial}{\partial x}]\overline{m}_{1n}(s,x) = \delta(x) +$$

$$(1-\delta_{n,N-1})\,(n+1)\mu\overline{m}_{1,n+1}(s,x) + (N-n)\lambda\overline{m}_{1,n-1}(s,x), \tag{3.9}$$

$$0 \le n \le N-1,$$

$$\overline{m}(s) = \int_0^\infty \overline{m}_{10}(s,x)\gamma(x)dx + \mu\overline{m}_{01}(s). \tag{3.10}$$

On the basis of these equations we prove the following Proposition.

Proposition 2. *The mean number of successful calls made during the busy period, $E\left[N^{SBP}\right]$, can be calculated by the formulas*

$$E\left[N^{SBP}\right] = \sum_{k=1}^{\infty} kP\left\{N^{SBP} = k\right\} = \overline{m}(0) =$$

$$\sum_{k=0}^{N-1} \overline{q}_{1k}(0,0)\left[1 + g_k(0) - \frac{k(\lambda+\mu)}{(N-1)\lambda}\right] +$$

$$\sum_{k=0}^{N-1} g_k(0)\overline{y}_k^{(0)} - \frac{\mu}{(N-1)\lambda}\overline{P}_{01}(0), \tag{3.11}$$

where $\overline{q}_{1k}(0,0)$ *satisfy the equations*

$$\sum_{k=0}^{N-1} \overline{q}_{1k}(0,0) \left\{ y_n^{(k)} \left[\delta_{n,N-1} + \frac{n\mu}{(N-n)\lambda} - g_k \right] - \right.$$

$$\left. (1-\delta_{n,N-1}) \frac{k(\lambda+\mu)}{(N-n-1)\lambda} \left(y_0^{(k)} + \cdots + y_{n-1}^{(k)} \right) \right\} = \sum_{k=0}^{N-1} g_k y_n^{(k)} \overline{y}_k^{(0)} +$$

$$\left[1 + \frac{n\mu}{(N-n)\lambda} \right] \overline{P}_{0n}(0) - (1-\delta_{n,N-1}) \frac{(n+1)\mu}{(N-n-1)\lambda} \overline{P}_{0,n+1}(0). \tag{3.12}$$

$$\sum_{k=0}^{N-1} \overline{q}_{1k}(0,0) g_k'(0) = E[\zeta] - \sum_{k=0}^{N-1} \overline{y}_k^{(0)} g_k'(0). \tag{3.13}$$

Here $g_k'(0)$ *is the derivative of the Laplace–Stieltjes transform,* $g(s)$ *of the service times in the point* $s = k(\lambda+\mu)$.

Proof. Similarly to the proof of Theorem 1 we apply the discrete transformations

$$\overline{m}_{1n}(s,x) = \sum_{k=0}^{N-1} y_n^{(k)} \overline{q}_{1k}(s,x),$$

solve equations (3.9) and express the quantities $\overline{m}_{1n}(s,x)$ in terms of the initial values $\overline{q}_{1k}(s,0)$:

$$\overline{m}_{1n}^{(1)}(s,x) = [1-G(x)] \sum_{k=0}^{N-1} y_n^{(k)} e^{-[k(\lambda+\mu)+s]x} \left[\overline{q}_{1k}^{(1)}(s,0) + \overline{y}_k^{(0)} \right],$$

$$0 \le n \le N-1.$$

Then from (3.8) we express the quantities $\overline{m}_{0n}(s)$,

$$\overline{m}_{0n}^{(1)}(s) = \frac{1}{(N-n)\lambda} \left[\sum_{k=0}^{N-1} y_n^{(k)} \overline{q}_{1k}^{(1)}(s,0) - \overline{P}_{0n}(s) \right], 1 \le n \le N-1$$

and substituting in (3.7) derive equations for the initial values $\overline{q}_{1k}^{(1)}(s,0)$:

$$\sum_{k=0}^{N-1} \overline{q}_{1k}^{(1)} \left\{ y_n^{(k)} \left[1 + \frac{n\mu + s}{(N-n)\lambda} - g_k \right] - \right.$$

$$\left.(1-\delta_{n,N-1})\,\frac{(n+1)\mu}{(N-n-1)\lambda}y_{n+1}^{(k)}\right\} = \sum_{k=0}^{N-1} g_k y_n^{(k)}\overline{y}_k^{(0)} +$$

$$\left(1+\frac{n\mu+s}{(N-n)\lambda}\right)\overline{P}_{0n}(s) - (1-\delta_{n,N-1})\,\frac{(n+1)\mu}{(N-n-1)\lambda}\overline{P}_{0,n+1}(s).$$

According to formulas (2.12) for $y_{n+1}^{(k)}$,the last equations are

$$\sum_{k=0}^{N-1}\overline{q}_{1k}(s,0)\left\{y_n^{(k)}\left[\delta_{n,N-1}+\frac{n\mu+s}{(N-n)\lambda}-g_k\right]-\right.$$

$$\left.(1-\delta_{n,N-1})\,\frac{k(\lambda+\mu)}{(N-n-1)\lambda}\left(y_0^{(k)}+\cdots+y_{n-1}^{(k)}\right)\right\} = \sum_{k=0}^{N-1} g_k y_n^{(k)}\overline{y}_k^{(0)} +$$

$$\left[1+\frac{n\mu+s}{(N-n)\lambda}\right]\overline{P}_{0n}(s) - (1-\delta_{n,N-1})\,\frac{(n+1)\mu}{(N-n-1)\lambda}\overline{P}_{0,n+1}(s),$$

which for $s=0$ give (3.12). Further, from (3.10) we have

$$\overline{m}(s) = \sum_{k=0}^{N-1}\overline{q}_{1k}(s,0)\left[y_0^{(k)}g_k(s)+\frac{\mu}{(N-1)\lambda}y_1^{(k)}\right]+$$

$$\sum_{k=0}^{N-1} y_0^{(k)}g_k(s)\overline{y}_k^{(0)} - \frac{\mu}{(N-1)\lambda}\overline{P}_{01}(s)$$

and substituting $y_0^{(k)}$ and $y_1^{(k)}$ according to (2.12):

$$\overline{m}(s) = \sum_{k=0}^{N-1}\overline{q}_{1k}(s,0)\left[1+g_k(s)-\frac{k(\lambda+\mu)}{(N-1)\lambda}\right]+$$

$$\sum_{k=0}^{N-1} g_k(s)\overline{y}_k^{(0)} - \frac{\mu}{(N-1)\lambda}\overline{P}_{01}(s). \tag{3.14}$$

For $s=0$ this equation gives (3.11). To prove the normalizing condition (3.13) we sum up equations (3.5) over $k=1,2,\ldots,$

$$\frac{\eta(0)-\eta(s)}{s} = \frac{\overline{m}(0)-\overline{m}(s)}{s},$$

then substitute according to (3.14) and obtain (3.13). This completes the proof of the proposition $\square$

Thus, to calculate the mean value of the number of successful calls made during the busy period, we need the mean busy period length, $E[\zeta]$ and the Laplace transforms $\overline{P}_{0n}(s)$ of the probabilities $P_{0n}(t)$. They can be calculated according to the formulas of Theorem 1. Further to solve this system we proceed in the same way as when solving the system (2.5)–(2.9).

In a similar way, multiplying each of equations (3.1)–(3.4) by k^2, each of (3.5) by k and summing over $k = 1, 2, \dots$ we obtain a system for the second moments of the joint distributions of the busy period length and the number of successful calls. With their help we can calculate the second moment of the number of successful calls, but the formulas are more complicated.

Figure 2 has the same structure as Figure 1 and shows the dependence of the mean number of successful calls made during the busy period against each of the system parameters. We see that the influence of the service distribution type is stronger that the one shown in Figure 1, especially on the behaviour against the source activation rate, μ (the blocking parameter).

4. Conclusion. In this paper we consider a finite source queueing system of M/G/1 type in which the failed customers are not allowed neither to queue nor to do repetitions. Instead, they are temporarily blocked in the orbit of inactive customers. We investigate descriptors of the system functioning, connected with its busy period: the busy period length and the number of successful calls made during the busy period. Formulas for computing the Laplace–Stieltjes transform of the busy period distribution, as well as the mean busy period length are derived. We also have obtained formulas for successively computing the distribution of the number of successful calls and its mean value. Some numerical examples are presented. This investigation can be continued by finding formulas for the second and third moments of the considered descriptions. Another interesting problem for eventual future work is to study the number of lost calls made during the busy period.

REFERENCES

[1] Amador J. On the distribution of the successful and blocked events in retrial queues with finite number of sources. In: Proceedings of the 5th International Conference on Queueing Theory and Network Applications, 2010, 15–22.

[2] Artalejo J. R., A. Gomez-Corral. Retrial Queueing Systems: A Computational Approach. Heidelberg: Springer, Berlin, 2008.

[3] Dragieva V. I. Steady state analysis of the M/G/1//N queue with orbit of blocked customers. *Annals of operations research*, Submitted.

[4] Falin G. I., J. R. Artalejo. A finite source retrial queue. *European Journal of Operational Research*, **108** (1998), 409–424.

[5] Falin G. I., J. G. C. Templeton. Retrial Queues. Chapman and Hall, London, 1997.

[6] Jaiswal N. Priority queues. Academic press, New York, 1969.

[7] Moscholios L., M. Logothetis. Engset multi - rate state - dependent loss models with Qos guarantee. *International Journal of Communications Systems*, **19** (2006), No 1, 67–93.

[8] Wang J., L. Zhao, F. Zhang. Analysis of the finite source retrial queues with server breakdowns and repairs. *Journal of Industrial and Management Optimization*, **7** (2011), 655–676.

[9] Wong E. W. M., A. Zalesky, M. Zukerman. On generalization of the Engset model. *IEEE Communications Letters*, **11** (2007), No 4, 360–362.

[10] Zhang J., Y. Peng, E. W. M. Wong, M. Zukerman. Sensitivity of blocking probability in the generalized Engset model for OBS. *IEEE Communications Letters*, **15** (2011), No 1, 1243–1245.

Velika Dragieva
Department of Mathematics
Sofia University of Forestry
10 Kliment Ohridski blvd.
Sofia, Bulgaria
e-mail: `dragievav@yahoo.com`

Received December 4, 2014
Final Accepted February 23, 2015