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GENERALIZED PRIORITY SYSTEMS. ANALYTICAL RESULTS AND NUMERICAL ALGORITHMS*

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Abstract. A class of priority systems with non-zero switching times, referred as generalized priority systems, is considered. Analytical results regarding the distribution of busy periods, queue lengths and various auxiliary characteristics are presented. These results can be viewed as generalizations of the Kendall functional equation and the Pollaczek-Khintchin transform equation, respectively. Numerical algorithms for systems' busy periods and traffic coefficients are developed.

1. Introduction. Concepts of Generalized Priority Systems.

Models of Queueing Systems play an important role in the analysis and modeling of various modern networks [1],[2]. Below we will present and discuss some results regarding a class of priority queueing systems with non-zero switching time, referred to as generalized priority systems [3]. This class of systems appeared as a

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result of mathematical formalization and consideration of the switchover times between priority classes and strategies in the free states. The assumption of nonzero switching of the service process allows one to take into consideration the various time losses existing in real time systems. Its consideration and analysis is very important from the applied point of view. From the theoretical point of view the study of the generalized priority systems correspond to the intrinsic logic of the development of Queueing Systems Theory. The results regarding generalized models are, naturally, more general and contain as particular cases many of the results already obtained in classical Queueing Theory. In what follows we present some examples.

Let's consider a queueing system $M_r|G_r|1$ with r priority classes of messages (requests). The priority classes are numbered in decreasing number of priorities, namely, it is assumed that i-messages have a higher priority than j-messages if $i < j$. It is also assumed that the server needs some time for switching the service process from the queue $i$to queue j. The length of i, j-switching is considered a random variable with distribution function $C_{ij}(x)$, $1 \leq i, j \leq n$, $i \neq j$. More details regarding the classification and nomenclature of such models are presented in [3], [5]. In what follows consider the preemptive priority discipline: the incoming message of the higher priority interrupt both the servicing and the switching. Regarding the future evolution of the interrupted servicing and switching only one preemptive scheme will be considered: P12 "resume", "repeat again" (the interrupted message will be served again). Also suppose that the switching C_{ij} depends only on index j, $C_{ij} = C_j$, and the strategy in the free state is "reset" [3].

2. System's Busy Period.

2. System's Busy Period. Denote by $\Pi_k(x)$ the distribution function (d.f.) of the busy period with the messages of priority no less than k, $\sigma_k = \lambda_1 + \cdots + \lambda_k$; $\beta_i(s)$, $c_j(s)$, $\pi(s), \ldots, \pi_k(s)$ are the Laplace-Stieltjes transforms of the distribution functions of $B_i(x)$, $C_j(x)$, $\Pi(x)$, $\ldots$, $\Pi_k(x)$, respectively.

Statement 1. *The Laplace-Stieltjes transform $\pi(s) = \pi_r(s)$ of the d.f. of the busy period is determined (at $k = r$) from the system of recurrent functional equations:*

$$\sigma_k\pi_k(s) = \sigma_{k-1}\pi_{k-1}(s+\lambda_k) + \sigma_{k-1}\{\pi_{k-1}(s+\lambda_k[1-\overline{\pi}_k(s)])- \\ - \pi_{k-1}(s+\lambda_k)\}\nu_k(s+\lambda_k[1-\overline{\pi}_k(s)]) + \lambda_k\pi_{kk}(s), \tag{1}$$

$$\pi_{kk}(s) = \nu_k(s+\lambda_k[1-\overline{\pi}_k(s)])\overline{\pi}_k(s), \tag{2}$$

$$\overline{\pi}_k(s) = h_k(s + \lambda_k[1 - \overline{\pi}_k(s)]), \tag{3}$$

where

$$\nu_k(s) = c_k(s + \sigma_{k-1}[1 - \pi_{k-1}(s)]), \tag{4}$$

$$h_k(s) = \beta_k(s + \sigma_{k-1})\left\{1 - \frac{\sigma_{k-1}}{s + \sigma_{k-1}}[1 - \beta_k(s + \sigma_{k-1})]\pi_{k-1}(s)\nu_k(s)\right\}^{-1} \tag{5}$$

Remark 1. The functions $\nu_k(s)$, $h_k(s)$ and $\pi_{kk}(s)$ can be viewed as auxiliary functions yet they all have a clear informative meaning. Thus, $\nu_k(s)$ and $h_k(s)$ are Laplace-Stieltjes transforms of the complete switching to priority class k and complete service time of k message, respectively.

Remark 2. *Gnedenko system's busy period.*

If $C_j = 0$, $j = 1, \ldots, r$, $r > 1$ from (1)–(5) follow the result published by Gnedenko et al. in [4]

$$\sigma_k \pi_k(s) = \sigma_{k-1}\pi_{k-1}(s + \lambda_k(1 - \pi_{kk}(s))) + \lambda_k \pi_{kk}(s)$$

$$\pi_{kk}(s) = h_k(s + \lambda_k(1 - \pi_{kk}(s)))$$

$$h_k(s) = \beta_k(s + \sigma_{k-1})\left\{1 - \frac{\sigma_{k-1}}{s + \sigma_{k-1}}[1 - \beta_k(s + \sigma_{k-1})]\pi_{k-1}(s)\right\}^{-1}$$

Remark 3. *Kendall–Takacs equation.*

If $C_j = 0$, $r = 1$ the system (1)–(5) represents a single equation

$$\pi_{11}(s) = h_1(s + \lambda_1(1 - \pi_{11}(s))).$$

But if $r = 1$ results that $h_1(s) = \beta_1(s)$ and $\pi_{11}(s) = \pi(s)$. Considering $\lambda_1 = \lambda$ and $\beta_1 = \beta$ the following equation holds: (known as Kendall–Takacs (1953) functional equation for the busy period for $M|G|1$):

$$\pi(s) = \beta(s + \lambda - \lambda\pi(s))$$

Thus, system (1)–(5) can be considered as an n-dimensional analog (in the sense of priority classes) of Kendall-Takacs equation.

3. Steady State Condition and Traffic Coefficients.

Statement 2. *Let* $\rho_k = \sum_{i=1}^{k} \lambda_i b_i$*, where*

$$b_1 = \frac{\beta_1 1 + c_{11}}{1 + \lambda_1 c_{11}}$$

$$b_k = \Phi_1 \dots \Phi_{k-1} \beta_{k1}(1 + \sigma_{i-1} c_{i1})$$

$$\Phi_1 = 1, \Phi_1 = 1 + (\sigma_i - \sigma_{i-1}\pi_{i-1}(\lambda_i))c_{i1}, \quad i = 2, \dots, k$$

If

(6) $$\rho_k < 1$$

then

(7) $$\sigma_k \pi_{k1} = \frac{\Phi_2 \dots \Phi_k + \rho k - 1}{1 - \rho_k}, \quad \bar{\pi}_{k1} = \frac{b_k}{1 - \rho_k}$$

(8) $$h_{k1} = \frac{b_k}{1 - \rho_{k-1}}, \quad \nu_{k1} = \frac{\Phi_2 \dots \Phi_{k-1}}{1 - \rho_{k-1}} c_{k1}$$

4. Probabilities of the System's State.

4.1. Probabilities of the $\rightarrow j$ **state.** Let ${}^{\rightarrow}P_j(t)$ denote the probability that at instant t the server is busy with orientation for servicing of a request of the j $(j = 1, \dots, r)$ priority class and ${}^{\rightarrow}p_j(s) = \int_0^{\infty} e^{-st\rightarrow} P_j(t)dt$—the Laplace transform of the ${}^{\rightarrow}P_j(t)$.

Statement 3.

$${}^{\rightarrow}p_j(s) = \frac{\sigma_j^{\rightarrow}\pi(s)}{s + \sigma - \sigma\pi(s)}$$

where

$$\sigma_{kj}^{\rightarrow}\pi_k(s) = \left\{ \psi_j(s)\gamma_{j-1}(s) + \frac{G_j(s)\sigma_{j-1}\pi_{j-1}(s)\psi_j(s)Q_j(s)}{1 - h_j(s)} \right\} \times$$

$$\times \prod_{i=j+1}^{k} \left\{ 1 + \psi_i(s)\gamma_{i-1}(s) + \frac{[1 + \sigma_{i-1}\pi_{i-1}(s)\psi_i(s)]\, G_i(s)Q_i(s)}{1 - h_i(s)} \right\}$$

$$\sigma_{\overrightarrow{kk}}\pi_k(s) = \psi_k(s)\gamma_{k-1}(s) + \frac{G_k(s)\sigma_{k-1}\pi_{k-1}(s)\psi_k(s)Q_k(s)}{1 - h_k(s)}, \quad \textit{for } j = k$$

where

$$Q_j(s) = \gamma_{j-1}(s)\nu_j(s) + \sigma_{j-1}\pi_{j-1}(s) + \lambda_j) - \sigma_j\pi_j(s)$$

$$\gamma_{i-1}(s) = \sigma_{i-1}\left[\pi_{i-1}(s) - \pi_{i-1}(s + \lambda_i)\right] + \lambda_i, \; \psi_j(s) = \frac{1 - c_j(s + \sigma_{j-1}\left[1 - \pi_{j-1}(s)\right])}{s + \sigma_{j-1}\left[1 - \pi_{j-1}(s)\right]}$$

$$G_j(s) = \frac{1 - \beta_j(s + \sigma_{j-1})}{s + \sigma_{j-1}\left[1 - \beta_j(s)(s + \sigma_{j-1})\right]\pi_{j-1}(s)\nu_j(s)}$$

4.2. Probabilities of the $^*j$ state. Let $^*P_j(t)$ denote the probability that at instant t the server is busy with servicing the requests of the class j $(j = 1, \ldots, r)$.

Statement 4.

$$^*p_j(s) = \frac{\sigma_j^*\pi(s)}{s + \sigma - \sigma\pi(s)}$$

where

$$\sigma_{kj}^*\pi_k(s) = \frac{G_j(s)\psi_j(s)Q_j(s)}{1 - h_j(s)} \times$$
$$\times \prod_{i=j+1}^{k} \left\{ 1 + \psi_i(s)\gamma_{i-1}(s) + \frac{G_i(s)\left[1 + \sigma_{i-1}\pi_(i-1)(s)\psi_i(s)\right] Q_i(s)}{1 - h_i(s)} \right\}, \text{ for } j < k$$

$$\sigma_{kk}^*\pi_k(s) = \frac{G_k(s)\psi_k(s)Q_k(s)}{1 - h_k(s)}, \quad \text{for } j = k.$$

Functions $Q_j(s), \ldots, G_j(s)$ have been determined above.

4.3. Steady State Probabilities. Let ${}^*\mathbb{P}_j$ and ${}^{\to}\mathbb{P}_j$ be the stationary probabilities of the ${}^*j$ and ${}^{\to}j$ state. If condition (6) is satisfied then

$$ {}^{\to}\mathbb{P}_j + \frac{\sigma_j^{\to}\pi(0)}{1+\sigma\pi_1} \quad {}^*\mathbb{P}_j = \frac{\sigma_j^*\pi(0)}{1+\sigma\pi_1} $$

where $\pi_1 = \pi_{r1}$.

Remark 4. *Danielean formula (1969) and free state probability* $\mathbb{P}_0$.

If $C_j = 0$, $r > 1$, preemptive priority discipline.

$$ {}^*\mathbb{P}_j = \mathbb{P}_j = \rho_j - \rho_{j-1}, $$

where

$$ \rho_j = \frac{1}{\sigma_{i-1}}\left[\frac{1}{\beta_i(\sigma_{i-1})} - 1\right] $$

$$ \mathbb{P}_0 = 1 - \sum_{i-1}^{r} \mathbb{P}_i = 1 - \rho_r. $$

5. Distribution of the Queue Length. Let $P_m(t)$ be the probability that at instant t there are $m = (m_1, \dots, m_r)$ requests in the system, where m_i is the number of requests of the i $(i = 1, \dots, r)$ class. Denote by $P(z,t) = \sum_{m\geq 1} P_m(t) z^m$, where $z^m = z_1^{m_1} \dots z_r^{m_r}$, $z = (z_1, \dots, z_r)$, $0 \leq z_i \leq 1$; $\rho(z,s) = \int_0^\infty e^{-st} P(z,t)dt$; $[]_k = \lambda_k(1-z_k) + \ldots + \lambda_r(1-z_r)$.

Statement 5. $p(z,s) = (1+\sigma\pi(z,s))/(s+\sigma-\sigma\pi(s))$, $\sigma\pi(z,s) = \sigma_r\pi_r(z,s)$ *– is determined from the recurrent relation:*

$$ \sigma_k\pi_k(z,s) = \sigma_{k-1}\pi_{k-1}(z,s) + \gamma_{k-1}(s,z)\nu_k(z,s) + \frac{h_k(z,s)}{z_k - h_k(s+[]_k)} \times $$
$$ \times \left\{\gamma_{k-1}(s,z)\nu_k(s+[]_k) + \sigma_{k-1}\pi_{k-1}(s+\lambda_k) - \sigma_k\pi_k(s)\right\} $$

where

$$ \gamma_{k-1}(s,z) = \sigma_{k-1}\left\{\pi_{k-1}(s+[]_k) - \pi_{k-1}(s+\lambda_k) - \sigma_k\pi_k(s)\right\} $$

$$h_k(z,s) = \{z_k\,[1-\beta_k(s+[]_k+\sigma_{k-1})]\times$$
$$\times\,[1+\sigma_{k-1}\,[\pi_{k-1}(z,s)+\pi_{k-1}(s+[]_k)\nu_k(z,s)]]\}\times$$
$$\times\,\{s+[]_k+\sigma_{k-1}-\sigma_{k-1}\,[1-\beta_k(s+[]_k+\sigma_{k-1})]\,\pi_{k-1}(s+[]_k)\nu_k(s+[]_k)\}^{-1}$$

$$\nu_k(z,s) = \frac{1-c_k(s+[]_k+\sigma_{k-1}\,[1-\pi_{k-1}(s+[]_k)])}{s+[]_k+\sigma_{k-1}\,[1-\pi_{k-1}(s+[]_k)]}\,[1+\sigma_{k-1}\pi_{k-1}(z,s)]$$

Functions $\nu_k(\cdot)$, $h_k(\cdot)$, $\pi_k(\cdot)$, $\pi(\cdot)$, are determined from (1)–(5).

Remark 5. *Steady state queue length distribution.*

Let $P(z)$ be the generating function of the queue length distribution in stationary state. If condition (6) is satisfied, then

$$P(z) = \lim_{s\uparrow 0} sp(z,s)$$

and

$$P(z) = \frac{1+\sigma\hat{\pi}(z)}{1+\sigma\pi_1} \tag{9}$$

where $\sigma\hat{\pi}(s) = \sigma_r\pi_r(z,0)$ and $\pi_1 = \pi_{r1}$.

Remark 6. *Pollaczek-Khinchin formula.*

If $C_j = 0$ and $r = 1$, from (8) follows:

$$P(z) = \frac{\beta(\lambda - z\lambda)(z-1)(1-\lambda\beta_1)}{z-\beta(\lambda - z\lambda)}$$

known as Pollaczek–Khinchin formula.

6. Numerical Algorithms.

6.1. Algorithm P12 for Busy Period. In what follows we present the algorithm for numerical solution of the k-busy periods $\pi_k(s)$ as well as the k-cycle of switching $\nu_k(s)$, the k-cycle of service $h_k(s)$ and kk-periods $\pi_{kk}(s)$.

Input: r, s^*, $\varepsilon > 0$, $\{\lambda_k\}_{k=1}^r$, $\{\beta_k(s)\}_{k=1}^r$, $\{c_k(s)\}_{k=1}^r$;

Output: $\pi_k(s^*)$;

Description:

IF (k==0) THEN $\pi_0(s^*) := 0$; *RETURN*

$k := 1$; $q := 1$; $\Lambda_0 := 0$;

Repeat $inc(q)$;

$\Lambda_q := \Lambda_{q-1} + \Lambda_q$;

Until $q == r$;

Repeat $\nu_k(s) := c_k(s^* + \Lambda_{k-1})[1 - \pi_{k-1}(s^*)]$);

$h_k(s^*) := \beta_k(s + \Lambda_{k-1}) +$

$$+\left\{1 - \frac{\Lambda_{k-1}}{s^* + \Lambda_{k-1}}\,[1 - \beta_k(s^* + \Lambda_{k-1})] \cdot \pi_{k-1}(s^*)\nu_k(s^*)\right\}^{-1};$$

$\pi_{kk}^{(0)}(s^*) := 0$; $n := 1$;

Repeat $\pi_{kk}^{(n)}(s^*) := h_k(s^* + \lambda_k - \lambda_k \pi_{kk}^{(n-1)}(s^*))$; $inc(n)$;

Until $\left|\pi_{kk}^{(n)}(s^*) - \pi_{kk}^{(n)}(s^*)\right| < \varepsilon$;

$$\pi_k(s^*) := \frac{\Lambda_{k-1}\pi_{k-1}(s^* + \lambda_k)}{\Lambda_k} + \frac{\Lambda_{k-1}}{\Lambda_k}(\pi_{k-1}(s^* + \lambda_k - \lambda_k \pi_{kk}(s^*)) -$$

$$-\pi_{k-1}(s^* + \lambda_k))\nu_k(s^* + \lambda_k[1 - \pi_{kk}(s^*)]) + \frac{\lambda_k}{\Lambda_k}\nu(s^* + \lambda_k - \lambda_k \pi_{kk}(s^*))\pi_{kk}(s^*);$$

$inc(k)$;

Until $k == r$;

End of Algorithm P12.

6.2. Algorithm Tr P12 for Traffic Evaluating. The following algorithm gives the numerical solution of the traffic coefficient ρ_k.

Input: r, s^*, $\varepsilon > 0$, $\{a_k\}_{k=1}^r$, $\{\beta_k(s)\}_{k=1}^r$, $\{c_k(s)\}_{k=1}^r$;

Output: $\pi_k(s^*)$, $\nu_k(s^*)$, $h_k(s^*)$, ρ;

Description:

IF (k==0) THEN $\pi_0(s^*) := 0$; *RETURN*

$k := 1$; $q := 1$; $\sigma_0 := 0$; $\rho := 1$;

$f_1 := 1$; $p := 1$;

$b_1 := (\beta_{11} + c_{11})/(1 + a_1 c_{11})$;

$\rho := a_1 b_1$;

Repeat $inc(q)$; $\sigma_q := \sigma_{q-1} + a_q$;

Until $q == r$;

Repeat

$\nu_k(s^*) := c_k(s + \sigma_{k-1}[1 - \pi_{k-1}(s)]);$

$h_k(s^*) := \beta_k(s^* + \sigma_{k-1}) \left\{ 1 - \frac{\sigma_{k-1}}{s^* + \sigma_{k-1}} [1 - \beta_k(s^* + \sigma_{k-1})] \cdot \pi_{k-1}(s^*)\nu_k(s^*) \right\}^{-1};$

$n := 1;\ \underline{\pi}_{kk}^{(n)}(0) := 0;\ \overline{\pi}_{kk}^{(n)}(0) := 1;$

Repeat

$\overline{\pi}_{kk}^{(n)}(s^*) := h_k(s^* + a_k - a_k\overline{\pi}_{kk}^{(n-1)}(s^*));$

$inc(n);$

Until $\frac{\overline{\pi}_{kk}^{(n)}(s^*) - \underline{\pi}_{kk}^{(n)}(s^*)}{2} < \varepsilon;$

$\pi_{kk}(s^*) := \frac{\overline{\pi}_{kk}^{(n)}(s^*) + \underline{\pi}_{kk}^{(n)}(s^*)}{2};$

$\pi_k(s^*) := \frac{\sigma_{k-1}\pi_{k-1}(s^* + a_k)}{\sigma_k} + \frac{\sigma_{k-1}}{\sigma_k}(\pi_{k-1}(s^*) + a_k -$
$- a_k\pi_{kk}(s^*)) - \pi_{k-1}(s^* + a_k)\nu_k(s^* + a_k[1 - \pi_{kk}(s^*)]) +$
$+ \frac{a_k}{\sigma_k}\nu(s^* + a_k - a_k\pi_{kk}(s^*))\pi_{kk}(s^*);$

$b_k := p \cdot \frac{1 + \sigma_{k-1}c_{k1}}{\sigma_{k-1}} \left(\frac{1}{\beta_k(\sigma_{k-1})} - 1 \right);$

$\rho := \rho + a_k b_k;$

$f_k := 1 + (\sigma_k - \sigma_{k-1}\pi_{k-1}(a_k))c_{k1};$

$p := f_k p;$

$inc(k);$

Until $k == r;$

End of Algorithm Tr P12.

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