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SYMBOLIC SOLVING OF PARTIAL DIFFERENTIAL EQUATION SYSTEMS AND COMPATIBILITY CONDITIONS*

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Abstract. An algorithm is produced for the symbolic solving of systems of partial differential equations by means of multivariate Laplace–Carson transform. A system of K equations with M as the greatest order of partial derivatives and right-hand parts of a special type is considered. Initial conditions are input. As a result of a Laplace–Carson transform of the system according to initial condition we obtain an algebraic system of equations. A method to obtain compatibility conditions is discussed.

1. Introduction. The Laplace transform has been useful in various problems of differential equations theory, including problems of partial equations (for example, [5, 8, 9, 14, 15, 16, 17, 19]). It serves as a basis for an operation calculus used in such applications. We must mention that kinds of operation calculus exist, such as Mikusinski-type operational calculus (see for example [2], [3], [20], [21]), or an approach by M. Gutterman (for example [22]), close to the ideas of J. Mikusinski, which use the convolution quotient, without referring to

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the Laplace transform. Our method permits not to reduce PDE equations to ordinary ones, but to obtain directly the algebraic system of equations for further activities with it.

On the other hand, there are many ways to use computer algebra systems for numerical or symbolic solving of PDE systems, for example the well-known use of MAPLE for a characteristics method that permits to simplify equations in many cases (for instance [18]).

The method produced in this paper is now of growing actuality because of the increasing relevance of symbolic computations nowadays. It reduces a system of partial differential equations with initial conditions to an algebraic linear system with polynomial coefficients which already encloses in a symbolic way the initial conditions requirements. An important advantage of the method is the establishment of compatibility conditions in a symbolic way for many types of PDE equations and systems of PDE equations. The type of PDE equations is not of great importance, nor is their order or the size of the system.

We produce an algorithm for symbolic solving of systems of linear partial differential equations by means of a multivariate Laplace–Carson transform. Systems of arbitrary number K of unknown functions and equations of arbitrary order M of derivatives are considered. The method allows not to reduce to canonical form the problem at the initial stage, it reduces it to solving a linear algebraic system with polynomial coefficients where efficient methods were developed (for example [6, 7, 10, 13]). A parallel algorithm may be constructed on the basis of the one produced. A way of parallelization is briefly discussed. So large systems of linear PDE may be solved in real time.

2. Problem statement. Denote $\widetilde{m} = (m_1, \ldots, m_n)$. Consider a system

$$\sum_{k=1}^{K} \sum_{m=0}^{M} \sum_{\widetilde{m}} a^j_{\widetilde{m}k} \frac{\partial^m}{\partial^{m_1} x_1 \ldots \partial^{m_n} x_n} u_k(x) = f_j(x), \tag{1}$$

where $j = 1, \ldots, K$, $u_k(x)$, $k = 1, \ldots, K$ are unknown functions of $x = (x_1, \ldots, x_n) \in \mathbf{R}^n_+$, $f_j \in S$, $a^j_{\widetilde{m}k}$ are real numbers, m is the order of a derivative, and k is the number of an unknown function. Here and further summing by $\widetilde{m} = (m_1, \ldots, m_n)$ is executed for $m_1 + \ldots + m_n = m$.

Functions $f_j(x)$ of the right-hand part of the system are in general composite. For each j consider $\mathbf{R}^n_+$ divided into parts $D^i_j = \Big\{x : x^{i_\nu}_{\nu,j} < x_\nu < x^{i_\nu+1}_{\nu,j}$, $\nu = 1, \ldots, n\}$, $i_\nu = 1, \ldots, I_{\nu,j}$, $x_{i_1} = 0, x_{I_{\nu,j}+1} = \infty$.

$$f_j(x) = f^i_j(x), \quad x \in D^i_j, \quad i = (i_1, \ldots, i_n).$$

Denote the set of i for which $i_\nu = 1, \dots, I_{\nu,j}$ by $\boldsymbol{I}_j$. As usually, $x - x_i = (x_1 - x_{i_1}, \dots, x_n - x_{i_n})$. $H(x) = H(x_1, \dots, x_n)$ is the multivariate Heaviside function which equals 1 only if $x_\nu \geq 0$ for all ν, and zero otherwise. For $l = (l_1, \dots, l_n)$ we denote $x^l = \prod_{\nu=1}^n x_\nu^{l_\nu}$, $\langle b_{js}^i, x \rangle = \sum_{\nu=1}^n b_{js}^{i,\nu} x_\nu$. Using Heaviside function, composite functions $f_j(x)$ may be written as follows

$$f_j(x) = \sum_{i \in \boldsymbol{I}_j} \left[\widetilde{f_j^i}(x - x_i) H(x - x_i) - \widetilde{f_j^i}(x - x_{i+1}) H(x - x_{i+1}) \right], \tag{2}$$

where $\widetilde{f_j^{I_j+1}}(x - x_{I_j+1}) = 0$, and

$$f_j^i(x) = \sum_{s=1}^{S_j^i} P_{js}^i(x) e^{\langle b_{js}^i, x \rangle}, \quad i = 1, \dots, I_j, \quad j = 1, \dots, k,$$

$P_{js}^i(x) = \sum_{l=0}^{L_{js}^i} c_{sl}^{ji} x^l$ is a polynomial, $L_{js}^i = (L_{js}^{i1}, \dots, L_{js}^{in})$. For a polynomial $P(x)$ we denote $\widetilde{P}(x - x_i)$ its expansion in powers of $(x_1 - x_{i_1}), \dots, (x_n - x_{i_n})$, and for $f(x) = P(x) e^{<a,x>}$ we denote $\widetilde{f}(x - x_i) = \widetilde{P}(x - x_i) e^{<a,x_i>} e^{<a,x-x_i>}$.

Denote by $\mathbf{A}$ a class of functions which are reducible to the form (2).

We solve a problem with initial conditions for each variable (not separating spatial or time data, it is defined by the particular problem). Introduce notations for them.

Denote by Γ^τ a set of vectors $\gamma = (\gamma_1, \dots, \gamma_n)$ such that $\gamma_\tau = 1$, $\gamma_i = 0$, if $i < \tau$, and γ_i equals 0 or 1 in all possible combinations for $i > \tau$. The number of elements in Γ^τ equals $2^{\tau-1}$.

Denote $\beta = (\beta_1, \dots, \beta_n)$, $\beta_i = 0, \dots, m_i$, a set of indexes such that the derivative of $u_k(x)$ of the order β_i with respect to the variables with numbers i equals $u_{\beta,\gamma}^k(x^{(\gamma)})$ at the point $x = x^\gamma$ with zeros at the positions μ for which the coordinates γ_μ of γ equal 1 (for example, if zeros stand only at the places with numbers $1, 2, 3$, then $\gamma = (1, 1, 1, 0, \dots, 0)$. We take functions $u_{\beta,\gamma}^k(x^{(\gamma)})$ also from the class $\mathbf{A}$.

Further we shall introduce a notion of compatible initial conditions and consider a method to obtain compatible initial conditions. The system (1) is to be solved under such conditions.

3. Laplace–Carson transform. Consider the space S of functions $f(x)$, $x = (x_1, \dots, x_n) \in \mathbf{R}_+^n$, $\mathbf{R}_+^n = \{x : x_i \geq 0, i = 1, \dots, n\}$, for which $\mathcal{M} > 0, a = (a_1, \dots, a_n) \in \mathbf{R}^n$, $a_i > 0$, $i = 1, \dots, n$, exist such that for all $x \in \mathbf{R}_+^n$ the following is true: $|f(x)| \leq \mathcal{M} e^{<a,x>}$. Evidently, $\mathbf{A} \subset S$.

On the space S the Laplace–Carson transform (LC) is defined as follows:

$$LC : f(x) \mapsto F(p) = p^1 \int_0^\infty e^{-<p,x>} f(x)dx,$$

$p = (p_1, \ldots, p_n)$, $p^1 = p_1 \ldots p_n$, $dx = dx_1 \ldots dx_n$. Integration is multivariate. The Laplace–Carson transform is more convenient than Laplace transform, as it transfers the Heaviside function into the unit, and it rather simplifies calculations.

The Laplace–Carson image of $(x - x_i)^l e^{<\alpha, x - x_i>} H(x - x_i)$ is

$$p^1 \prod_{\nu=1}^{n} \frac{l_\nu!}{(p_\nu - \alpha_\nu)^{l_\nu+1}} e^{-<p,x_i>}, \quad \alpha = (\alpha_1, \ldots, \alpha_n). \tag{3}$$

That is why LC is performed symbolically on the class **A**.

Let $LC : u_k \mapsto U_k, u^k_{\beta,\gamma}(x^{(\gamma)}) \mapsto U^k_{\beta,\gamma}(p^{(\gamma)}), f_j \mapsto F_j$. The notation $p^{(\gamma)}$ corresponds to the notation $x^{(\gamma)}$. Denote by $\|\gamma\|$ the "length" of γ — the number of units in γ.

The LC of the left-hand side of the system (1) is being written formally. Write at first the image of a derivative;

$$LC : \frac{\partial^m}{\partial^{m_1} x_1 \ldots \partial^{m_n} x_n} u_k(x) \;\mapsto$$

$$p^m U_k(p) \;+\; \sum_{\nu=1}^{n} \sum_{\beta_\nu=0}^{m_\nu} \sum_{\gamma \in \Gamma^\nu} (-1)^{\|\gamma\|} p_1^{m_1-\beta_1-\gamma_1} \ldots p_n^{m_n-\beta_n-\gamma_n} U^k_{\beta,\gamma}(p^{(\gamma)}).$$

Denote

$$\Phi^j_{mk} = \sum_{\widetilde{m}} a^j_{\widetilde{m}k} \sum_{\nu=1}^{n} \sum_{\beta_\nu=0}^{m_\nu} \sum_{\gamma \in \Gamma^\nu} (-1)^{\|\gamma\|} p_1^{m_1-\beta_1-\gamma_1} \ldots p_n^{m_n-\beta_n-\gamma_n} U^k_{\beta,\gamma}(p^{(\gamma)}).$$

As a result of a Laplace–Carson transform of the system (1) according to the initial conditions we obtain the algebraic system relative to U_k:

$$\sum_{k=1}^{K} \sum_{m=0}^{M} \sum_{\widetilde{m}} a^j_{\widetilde{m}k} p^{\widetilde{m}} U_k(p) = F_j - \sum_{k=1}^{K} \sum_{m=0}^{M} \Phi^j_{mk}, j = 1, \ldots, K. \tag{4}$$

4. Solution of the algebraic system. Efficient methods for solving such systems are developed according to size or density of the algebraic system (for example [10, 6, 13, 7]).

Denote by D the determinant of the system (4), by D_k the minor that is obtained by replacing the kth column by the column of the right-hand part of (4). The solution of the system (4) is

$$U_k = \frac{D_k}{D}. \tag{5}$$

5. The inverse Laplace–Carson transform and compatibility conditions. The next step of the algorithm is the inverse Laplace–Carson transform of U_k. If it is possible, then we obtain as result the solution of the input system, which is unique (due to uniqueness of the inverse LC) and satisfies the input initial conditions. According to the character of the system – it is linear with constant coefficients – the solutions depend continuously on changing of initial conditions inside the class $\mathbf{A}$. This means that we have found a correct (in a traditional meaning) solution of the system.

The inverse Laplace–Carson transform of U_k exists under the conditions of convergence of the Laplace integral of $\widetilde{U_k}(p) = \frac{1}{p^1}U_k(p)$, i. e., the following:

1^* $\sigma \in \mathbf{R^n}, \sigma = (\sigma_\mathbf{1}, \ldots, \sigma_\mathbf{n})$, exists, such that $\widetilde{U_k}$ is holomorphic in the domain $\mathrm{Re}p_\nu > \sigma_\nu$;

2^* $\lim_{p\to\infty} \widetilde{U_k}(p) = 0$;

3^* $\widetilde{U_k}(p)$ is integrable by each p_ν along any line $\mathrm{Re}\, p_\nu > \sigma_{\nu_0}$, $\sigma_{\nu_0} > \sigma_\nu$.

We consider functions from class $\mathbf{A}$. According to the properties of the LC transform, in the fractions (5) the degrees of the polynomials in the numerators are less than the degree of the denominator, and these fractions may be represented as linear combinations of expressions of type (3). So 2^* is fulfilled. 3^* is satisfied automatically as soon as 1^* takes place, due to the type (3) of expressions. So it remains to verify the implementation of 1^*.

Denote the set of zeros of D by $\mathcal{Q}$. In the case when $\mathcal{Q}$ has no infinite limit points at $\mathbf{P}^+ = \{p : \mathrm{Re}\, p_\nu > 0,\ \nu = 1, \ldots, n\}$ the conditions 1^*–3^* are satisfied for any initial conditions, the solution of (1) is correct.

Consider the opposite case: denote by $\mathcal{Q}_\infty$ the subset of $\mathcal{Q}$ with infinite limit points at $\mathbf{P}^+$. The condition 1^* fulfills if and only if functions $u^k_{\beta,\gamma}(x^{(\gamma)})$, $k = 1, \ldots, n$, exist and are being used as initial conditions, such that D_k has zeros at $\mathcal{Q}_\infty$ of multiplicity no less than the multiplicity of corresponding zeros of D.

This demand produces requirements to the LC images of initial conditions functions, and after the LC^{-1} transform, to the initial conditions. They turn out to be dependent.

Initial conditions whose LC images satisfy the conditions $1^* - 3^*$ *we call compatible.*

Thereby the following theorem is proved.

Theorem. *If $\mathcal{Q}$ has no infinite limit points at $\boldsymbol{P}^{+}$, the initial conditions assumed above are compatible. If $\mathcal{Q}$ has infinite limit points at $\boldsymbol{P}^{+}$ initial conditions are compatible if and only if their images satisfy the condition: D_k has zeros at $\boldsymbol{P}^{+}$ of multiplicity no less than the multiplicity of corresponding zeros of D.*

The theorem may be considered as the condition for the existence of the inverse Laplace–Carson transform of $U_k(p)$.

6. Independence of the type of equations. As can be seen from the presented constructions, the order or the type of equations (elliptic, hyperbolic, parabolic, etc.) does not play a significant role in the method of Laplace–Carson transform. Moreover, the method allows not to reduce (or to reduce to canonical form) the problem at the initial stage. However some special effects may occur in elliptic case.

7. Examples. In the Section 5 there was given the foundation of the method to obtain compatibility conditions, and this method was described in a general case. It would be useful not to overload the text with bulky common constructions of the implementation and to demonstrate technical details through rather transparent examples.

We present two examples: a system of the first order with composite right-hand parts and an equation of the fourth order.

Implemented in Mathematica.

7.1. Example 1. To demonstrate the techniques of the LC algorithm let us consider in detail the solving of a system of three equations with three unknown functions $f(x, y, z)$, $g(x, y, z)$, $h(x, y, z)$ on $\mathbf{R}^3_+$.

$$\begin{array}{rcl} \frac{\partial}{\partial x}f + \frac{\partial}{\partial z}g + \frac{\partial}{\partial y}h & = & \phi \\ \frac{\partial}{\partial z}f + \frac{\partial}{\partial x}g + \frac{\partial}{\partial y}h & = & \psi \\ \frac{\partial}{\partial y}f + \frac{\partial}{\partial x}g + \frac{\partial}{\partial z}h & = & \theta \end{array}$$

The functions ϕ, ψ, θ at the right-hand parts are composite. Using Heaviside function we write them in the following way:
$\phi = x(H(x, y, z) - H(x-1, y-1, z-1))$; $\psi = y(H(x, y, z) - H(x-1, y-1, z-1))$; $\theta = z(H(x, y, z) - H(x-1, y-1, z-1))$.

We shall consider the problem when the values of the unknown function at the zeros of x, y, z are taken as initial conditions.

If we have the derivatives of the first order with respect to each variable we need nine initial conditions – three for every unknown function – at $(0, y, z)$, $(x, 0, z)$, $(x, y, 0)$, correspondingly to the order of the derivative. A requirement is the coincidence of correspondent functions values at the intersection of these planes.

Denote the values of the functions at these points as follows:
$f(0, y, z) = f^x$, $f(x, 0, z) = f^y$, $f(x, y, 0) = f^z$, $g(0, y, z) = g^x$, $g(x, 0, z) = g^y$, $g(x, y, 0) = g^z$, $h(0, y, z) = h^x$, $h(x, 0, z) = h^y$, $h(x, y, 0) = h^z$.

Denote the images of the LC transform of f, g, h, respectively by u, v, w.

To be transparent in the example we denote the LC images of the initial conditions functions by the nine Greek letters $\alpha, \beta, \gamma, \delta, \tau, \varepsilon, \xi, \tau, \sigma$:

Table 1

	(q, r)	(p, r)	(p, q)
f	α	η	δ
g	ε	ξ	β
h	τ	γ	σ

In the table the first column displays the functions for which the LC images of the initial conditions are considered; the first line indicates the variables upon which these images depend. Note that in our system we have no the derivatives $\frac{\partial}{\partial y} g$ and $\frac{\partial}{\partial x} h$, so we do not need τ and ξ.

Applying the Laplace–Carson transform to the system (1) we obtain the algebraic system

$$pu + rv + qw - p\alpha - r\beta - q\gamma = F,$$
$$ru + pv + qw - r\delta - p\varepsilon - q\gamma = G,$$
$$qu + pv + rw - q\eta - p\varepsilon - r\sigma = H,$$

where
$F = -e^{-p-q-r} + \frac{1}{p} - \frac{1}{p}e^{-p-q-r}$; $G = -e^{-p-q-r} + \frac{1}{q} - \frac{1}{q}e^{-p-q-r}$; $H = -e^{-p-q-r} + \frac{1}{r} - \frac{1}{r}e^{-p-q-r}$.

The solution of this system is

$$u = -\frac{-pq^2 + pqr - pq^2r + qr^2 + pqr^2 + q^2r^2 - r^3 - qr^3}{qr(p-r)(q-r)(p+q+r)} +$$

$$e^{p+q+r}\left(\frac{pq^2 - pqr - qr^2 + r^3 + p^2q^2r\alpha - p^2qr^2\alpha + pq^2r^2\beta - pqr^3\beta}{qr(p-r)(q-r)(p+q+r)} + \right.$$

$$\frac{-pq^2r^2\gamma + q^2r^3\gamma - pq^2r^2\delta + qr^4\delta}{qr(p-r)(q-r)(p+q+r)} +$$

$$\left.\frac{-pq^2r^2\epsilon + pqr^3\epsilon + pq^3r\eta - q^3r^2\eta + pq^2r^2\sigma - q^2r^3\sigma}{qr(p-r)(q-r)(p+q+r)}\right)$$

$$v = \frac{-p^2q^2 - p^2q^2r + q^3r + p^2r^2 + p^2qr^2 + pq^2r^2 - qr^3 - pqr^3}{pqr(p-r)(q-r)(p+q+r)} -$$

$$e^{p+q+r}\left(\frac{p^2q^2 - q^3r - p^2r^2 + qr^3 - p^2q^3r\alpha + p^2qr^3\alpha - pq^3r^2\beta}{qr(p-r)(q-r)(p+q+r)} +\right.$$

$$\frac{pqr^4\beta - p^2q^2r^2\gamma + pq^2r^3\gamma + pq^3r^2\delta - p^2qr^3\delta + p^3q^2r\epsilon}{qr(p-r)(q-r)(p+q+r)} +$$

$$\left.\frac{p^2q^3r\epsilon - p^3qr^2\epsilon - p^2q^2r^2\epsilon + p^2q^3r\eta - pq^3r^2\eta + p^2q^2r^2\sigma - pq^2r^3\sigma}{qr(p-r)(q-r)(p+q+r)}\right),$$

$$w = \frac{p^2q - p^2r - q^2r - pq^2r + qr^2 + pqr^2 + q^2r^2 - qr^3}{pqr(p-r)(q-r)(p+q+r)} -$$

$$e^{p+q+r}\left(\frac{-p^2q + p^2r + q^2r - qr^2 + p^2q^2r\alpha - p^2qr^2\alpha}{qr(p-r)(q-r)(p+q+r)} +\right.$$

$$\frac{pq^2r^2\beta - pqr^3\beta + p^2q^2r\gamma + pq^3r\gamma - pq^2r^2\gamma - q^3r^2\gamma}{qr(p-r)(q-r)(p+q+r)} +$$

$$\left.\frac{p^2qr^2\delta - q^2r^3\delta - pq^2r^2\epsilon + pqr^3\epsilon - p^2q^2r\eta + q^2r^3\eta - p^2qr^2\sigma + qr^4\sigma}{qr(p-r)(q-r)(p+q+r)}\right),$$

The determinant D of the system equals

$$D = -(p-r)(q-r)(p+q+r).$$

The bracket $(p+q+r)$ is not important for solving the problem of compatibility—its zeros do not belong to $\mathcal{Q}_\infty$.

Consider the sets $p = r$, $q = r$. They form the set $\mathcal{Q}_\infty$. We demand that the numerators of the solutions be zero on these sets. To indicate that the functions of the initial conditions are taken for $p = r$ or $q = r$ we use the notations displaced in the following table. If for a function $p = r$ is set, we use this function with the index 1, if $q = r$ is set, we use this function with the index 2. To demonstrate the algorithm of getting compatibility conditions display initial conditions and their transformations after substituting of points of $\mathcal{Q}_\infty$ into the table.

Table 2

	$\alpha(q,r)$	$\varepsilon(q,r)$	$\tau(q,r)$	$\theta(p,r)$	$\xi(p,r)$	$\gamma(p,r)$	$\delta(p,q)$	$\beta(p,q)$	$\sigma(p,q)$
$p=r$	$\alpha(q,r)$	$\varepsilon(q,r)$	$\tau(q,r)$	$\theta_1(r,r)$	$\xi_1(r,r)$	$\gamma_1(r,r)$	$\delta_1(r,q)$	$\beta_1(r,q)$	$\sigma_1(r,q)$
$q=r$	$\alpha_2(q,r)$	$\varepsilon_2(r,r)$	$\tau_2(r,r)$	$\theta(p,r)$	$\xi(p,r)$	$\gamma(p,r)$	$\delta_2(p,r)$	$\beta_2(p,r)$	$\sigma_2(p,r)$

Substituting $p = r$ and $q = r$ into the numerators of u, v, w, we obtain a system of 6 equations, which connect functions $\alpha, \beta, \gamma, \delta, \delta, \ldots, \delta_2$.

$$\left\{\begin{array}{rcl}
-q^2r + 2qr^2 - r^3 + e^{q+2r}(q^2r - 2qr^2 + r^3 + q^2r^3\alpha - qr^4\alpha + q^2r^3\beta_1 & - & \\
qr^4\beta_1 - q^2r^3\delta_1 + qr^4\delta_1 - q^2r^3\epsilon + qr^4\epsilon) & = & 0 \\
q^3r - q^2r^2 - qr^3 + r^4 + e^{q+2r}(-q^3r + q^2r^2 + qr^3 - r^4 - q^3r^3\alpha + qr^5\alpha & - & \\
q^3r^3\beta_1 + qr^5\beta_1 + q^3r^3\delta_1 - qr^5\delta_1 + q^3r^3\epsilon - qr^5\epsilon) & = & 0 \\
-q^2r + 2qr^2 - r^3 + e^{q+2r}(q^2r - 2qr^2 + r^3 + q^2r^3\alpha - qr^4\alpha + q^2r^3\beta_1 & - & \\
qr^4\beta_1 - q^2r^3\delta_1 + qr^4\delta_1 - q^2r^3\epsilon + qr^4\epsilon) & = & 0 \\
-pr^4\gamma + r^5\gamma - pr^4\delta_2 + r^5\delta_2 + pr^4\eta & - & \\
r^5\eta + pr^4\sigma_2 - r^5\sigma_2 & = & 0 \\
-p^2r^4\gamma + pr^5\gamma - p^2r^4\delta_2 + pr^5\delta_2 + p^2r^4\eta & - & \\
pr^5\eta + p^2r^4\sigma_2 - pr^5\sigma_2 & = & 0 \\
p^2r^3\gamma - r^5\gamma + p^2r^3\delta_2 - r^5\delta_2 - p^2r^3\eta & + & \\
r^5\eta - p^2r^3\sigma_2 + r^5\sigma_2 & = & 0
\end{array}\right.$$

Solving it with respect to these variables, we get two conditions on them:

$$\begin{array}{rcl}
\alpha & = & -\frac{(1-e^{-q-2r})(q-r)}{qr^2} - \beta_1 + \delta_1 + \epsilon, \\
\gamma & = & -\delta_2 + \eta + \sigma_2.
\end{array} \tag{6}$$

We may take arbitrarily all images of initial conditions except of α and γ and obtain α and γ according to the conditions (6).

For example, we may take the following functions in Table 1.

Table 3

	(q,r)	(p,r)	(p,q)
f	$e^{-q-2r}(q-r)/(qr^2)-(q-2)/(qr^2)$	$1/(pr)$	$1/(p^2q)$
g	$1/(qr^2)$	ξ	$1/(pq)$
h	τ	$(p-r+pr)/(p^2r^2)$	$1/(pq^2)$

The corresponding initial conditions are the follows:
$f^x = \frac{1}{2}((-1+2y)z^2-(2y-z)(-2+z)H(-1+y)H(-2+z)),\ g^x = \frac{yz^2}{2}$, $f^y = xz, h^y = \frac{1}{2}(2xz - x^2z + xz^2),\ f^z = \frac{x^2y}{2},\ g^z = xy,\ h^z = \frac{xy^2}{2}$.

Substituting the functions $\alpha, \beta, \gamma, \ldots$ from Table 3 into the solution u, v, w, after inverse LC transform we obtain the solution of the system (1) corresponding to these initial conditions:

$$f = \frac{1}{24}\big(12x^2y - 24xyz + 24yz^2 - 8z^3 -$$
$$H(-y+z)(8(3xy+3y^2-y^3+3xz-3yz+3y^2z-3yz^2+z^3)+$$
$$8(y-z)(3x-3y+y^2-2yz+z^2)H(-x+y)-$$
$$12(y-z)(2+y-z)H(-1+x,-1+y)+$$
$$12(y-z)(2+y-z)H(-1+x,-x+y))-$$
$$3(-2+x+z)(-2+3x-4y+3z)H(-1+y,-2+x+z)-$$
$$(-9x^2+36xy-36y^2-18xz+36yz-9z^2)H(-1+y,x-2y+z)-$$
$$(-12-24x-9x^2+24y+12xy+6xz-12yz+3z^2)H(-1-x+y,-2-x+z)-$$
$$(9x^2-36xy+36y^2+18xz-36yz+9z^2)H(-1-x+y,x-2y+z)-$$
$$(-12x^2-8x^3+24xz+24x^2z-12z^2-24xz^2+8z^3)H(-x+y,-x+z)-$$
$$(12+24y-24z-24yz+12z^2)H(-1+x,-1+y,-1+z)-$$
$$(-12+24x-9x^2-24y+12xy+24z-18xz+12yz-9z^2)H(-1+x,-1+y,-2+x+z)-$$
$$(9x^2-36xy+36y^2+18xz-36yz+9z^2)H(-1+x,-1+y,x-2y+z)-$$
$$(-24x+9x^2-12xy+24z-6xz+12yz-3z^2)H(-1+x,-x+y,-x+z)+$$
$$9(x-2y+z)^2H(-1+x,-x+y,x-2y+z)\big)$$

$$g = \frac{1}{24}\left(12x^2y - 24xyz + 24yz^2 - 8z^3 -\right.$$
$$H(-y+z)(8(3xy + 3y^2 - y^3 + 3xz - 3yz + 3y^2z - 3yz^2 + z^3)+$$
$$8(y-z)(3x - 3y + y^2 - 2yz + z^2)H(-x+y)-$$
$$12(y-z)(2+y-z)H(-1+x, -1+y)+$$
$$12(y-z)(2+y-z)H(-1+x, -x+y))-$$
$$3(-2+x+z)(-2+3x-4y+3z)H(-1+y, -2+x+z)-$$
$$(-9x^2 + 36xy - 36y^2 - 18xz + 36yz - 9z^2)H(-1+y, x-2y+z)-$$
$$(-12 - 24x - 9x^2 + 24y + 12xy + 6xz - 12yz + 3z^2)H(-1-x+y, -2-x+z)-$$
$$(9x^2 - 36xy + 36y^2 + 18xz - 36yz + 9z^2)H(-1-x+y, x-2y+z)-$$
$$(-12x^2 - 8x^3 + 24xz + 24x^2z - 12z^2 - 24xz^2 + 8z^3)H(-x+y, -x+z)-$$
$$(12 + 24y - 24z - 24yz + 12z^2)H(-1+x, -1+y, -1+z)-$$
$$(-12+24x-9x^2-24y+12xy+24z-18xz+12yz-9z^2)H(-1+x, -1+y, -2+x+z)-$$
$$(9x^2 - 36xy + 36y^2 + 18xz - 36yz + 9z^2)H(-1+x, -1+y, x-2y+z)-$$
$$(-24x + 9x^2 - 12xy + 24z - 6xz + 12yz - 3z^2)H(-1+x, -x+y, -x+z)+$$
$$\left.9(x-2y+z)^2H(-1+x, -x+y, x-2y+z)\right)$$

$$h = 1/24\left(12xy^2 - 12x^2z - 24yz + 12xz^2 + 24yz^2 - 8z^3 +\right.$$
$$4(y-z)H(-y+z)(-2(3x - 3y + y^2 - 2yz + z^2)+$$
$$(2(3x - 3y + y^2 - 2yz + z^2) + 3(2+y-z)H(-1+x))H(-x+y)-$$
$$3(2+y-z)H(-1+x, -1+y)) + 12(-2+x+z)H(-1+y, -2+x+z)-$$
$$12y(-2+x+z)H(-1+y, -2+x+z) + 9(-2+x+z)^2H(-1+y, -2+x+z)-$$
$$9(x-2y+z)^2H(-1+y, x-2y+z) - 12(2+x-z)H(-1-x+y, -2-x+z)-$$
$$12x(2+x-z)H(-1-x+y, -2-x+z) + 12y(2+x-z)H(-1-x+y, -2-x+z)+$$
$$3(2+x-z)^2H(-1-x+y, -2-x+z) + 9(x-2y+z)^2H(-1-x+y, x-2y+z)-$$
$$12(x-z)^2H(-x+y, -x+z) - 8(x-z)^3H(-x+y, -x+z)-$$

$24(-1+z)H(-1+x,-1+y,-1+z)+12(-1+z)^2H(-1+x,-1+y,-1+z)-$
$12(-2+x+z)H(-1+x,-1+y,-2+x+z)+12y(-2+x+z)H(-1+x,-1+y,-2+x+z)-$
$9(-2+x+z)^2H(-1+x,-1+y,-2+x+z)+9(x-2y+z)^2H(-1+x,-1+y,x-2y+z)+$
$12x(x-z)H(-1+x,-x+y,-x+z)-3(x-z)^2H(-1+x,-x+y,-x+z)+$
$24(-x+z)H(-1+x,-x+y,-x+z)+12y(-x+z)H(-1+x,-x+y,-x+z)-$
$9(x-2y+z)^2H(-1+x,-x+y,x-2y+z)\big)$

7.2. Example 2. Consider the equation of forced vibration of an elastic rod:

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^4 f}{\partial y^4}=xy.$$

Initial conditions:

$f(0,y)=a(y);\quad \frac{\partial f(x,y)}{\partial x}\big|_{x=0}=b(y);$

$f(x,0)=c(x);\quad \frac{\partial f(x,y)}{\partial y}\big|_{y=0}=d(x);$

$\frac{\partial^2 f(x,y)}{\partial y^2}\big|_{y=0}=g(x);\quad \frac{\partial^3 f(x,y)}{\partial y^3}\big|_{y=0}=h(x)$

$$LC: f(x,y)\mapsto u(p,q),$$

$a(y)\mapsto\alpha(q),\quad b(y)\mapsto\beta(q),$
$c(x)\mapsto\gamma(p),\quad d(x)\mapsto\delta(p),$
$g(x)\mapsto\sigma(p),\quad h(x)\mapsto\tau(p).$

As a result of LC we obtain the algebraic equation:

$$p^2u-p^2\alpha-p\beta-qu+q\gamma qu+q\gamma=\frac{1}{pq}$$

$D=p^2+q^4$
Then

$$u=\frac{1+p^3q\alpha+p^2q\beta+pq^5\gamma+pq^4\delta+pq^3\sigma+pq^2\tau}{pq(p^2+q^4)};.$$

Q consists of two sets of zeros of the denominator $D=p^2+q^4$:
$p=iq^2$, $p=-iq^2$.

In these sets the numerator of u equals respectively

$$A_1 = 1 - iq^7\alpha - q^5\beta + iq^7\gamma_1 + iq^6\delta_1 + iq^5\sigma_1 + iq^4\tau_1;$$
$$A_2 = 1 + iq^7\alpha - q^5\beta - iq^7\gamma_2 - iq^6\delta_2 - iq^5\sigma_2 - iq^4\tau_2,$$

where

$\gamma_1,\ \delta_1,\ \sigma_1,\ \tau_1$ are the values of the functions $\gamma,\ \delta,\ \sigma,\ \tau$ at $p = iq^2$,
$\gamma_1,\ \delta_1,\ \sigma_1,\ \tau_1$ – at $p = iq^2$.

The functions with indexes 1 and 2 depend on different arguments iq^2 and $-iq^2$, respectively. So it is convenient to take the originals c, d, g, h of $\gamma,\ \delta,\ \sigma,\ \tau$ as data functions of initial conditions and to find a, b as compatible with them. Note that this is a characteristic speciality of equations of such type, for example of elliptic equations.

Solve

$$\begin{cases} A_1 &= 0, \\ A_2 &= 0 \end{cases}$$

with respect to α, β.

Compatibility conditions on images of LC:

$$\alpha = -\frac{-q^3\gamma_1 - q^3\gamma_2 - q^2\delta_1 - q^2\delta_2 - q\sigma_1 - q\sigma_2 - \tau_1 - \tau_2}{2q^3};$$

$$\beta = \frac{i(-2i + q^7\gamma_1 - q^7\gamma_2 + q^6\delta_1 - q^6\delta_2 + q^5\sigma_1 - q^5\sigma_2 + q^4\tau_1 - q^4\tau_2)}{2q^5}.$$

Taking the concrete functions $c(t), d(t), g(t), h(t)$ of initial conditions, we obtain $a(x)$ and $b(x)$ as compatible with them. In such way we may define, for example, the following compatible initial conditions:

$a = 1 - \frac{x^4}{12},\ b = \frac{x^5}{120}, c = 1 + t^2.$

Finally we obtain the solution satisfying the initial conditions:

$$f(t, x) = 1 + t^2 - \frac{x^4}{12} + \frac{tx^5}{120}.$$

8. A parallelization of solving. A parallel algorithm may be constructed on the basis of the one produced.

1. The Laplace transform of right-hand parts is independent on the number of equation, it is to be produced in parallel way.

2. The Laplace transform of left-hand parts is absolutely formal and also independent, it is to be produced in parallel way.

3. Solving the algebraic system permits various methods. The most effective ones are based on parallel computations.

4. The inverse Laplace transform for each obtained function is to be parallelized.

9. Conclusion. Let us adduce advantages of the algorithm presented in the paper.

1. The Laplace–Carson transform of a system with input functions from class **A** is being fulfilled in a symbolic way.

2. The algebraic system obtained after the Laplace transform may be solved by methods most convenient and efficient for each specific case.

Call a rational fraction *"a proper fraction"* if the degree of each variable (over **C**) in the numerator is less then its degree in the denominator.

Call class **B** a set of equations defined by these conditions:

– the solutions of algebraic system may be represented as sums of proper fractions with exponential coefficients,

– the denominators of these proper fractions may be reduced to a product of linear functions.

3. If a system of differential equations belongs to class **B**, then the inverse Laplace transform is being fulfilled symbolically.

4. The application of the Laplace–Carson transform permits obtaining compatibility conditions in a symbolic way.

5. The order of derivatives, the size of the systems and in many cases the types of equations are not significant for LC method.

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