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HAUSDORFF APPROXIMATION OF FUNCTIONS DIFFERENT FROM ZERO AT ONE POINT – IMPLEMENTATION IN PROGRAMMING ENVIRONMENT MATHEMATICA

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Abstract. Moduli for numerical finding of the polynomial of the best Hausdorff approximation of the functions which differs from zero at just one point or having one jump and partially constant in programming environment MATHEMATICA are proposed. They are tested for practically important functions and the results are graphically illustrated. These moduli can be used for scientific research as well in teaching process of Approximation theory and its application.

1. Definitions and Notations. (we keep to the notations used in [6]).

Let the bounded function f be defined on the interval Δ. We define by $F(f)$ the segment-valued function on Δ as follows

$$F(f;x) = [I(f;x), S(f;x)],$$

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where

$$I(f;x) = \lim_{\delta\to 0} \inf\{y \in f(t) : t \in [x-\delta, x+\delta] \cap \Delta\},$$

$$S(f;x) = \lim_{\delta\to 0} \sup\{y \in f(t) : t \in [x-\delta, x+\delta] \cap \Delta\}.$$

The Hausdorff distance with parameter $\alpha > 0$ between the bounded functions f and g defined on the interval Δ is

$$r(\alpha; f, g) = \max\{h(\alpha; f, g), h(\alpha; g, f)\},$$

where

$$h(\alpha; f, g) = \max_{(x,y)\in F(f)} \min_{(v,t)\in F(g)} \max\{\alpha^{-1}|x-v|, |y-t|\}.$$

If H_n denotes the set of all algebraic polynomials of degree less or equal n then the best Hausdorff approximation $E^r_{n,\alpha}(f)$ of the function f by polynomials from H_n is

$$E^r_{n,\alpha}(f) = \inf_{p\in H_n} r(\alpha; f, p),$$

and the polynomial $q \in H_n$ for which $E^r_{n,\alpha}(f) = r(\alpha; f, q)$ is called the polynomial of the best Hausdorff approximation of degree n.

From now on let $\Delta = [-1, 1]$ and let us consider the following functions and their best Hausdorff approximations which are proved in [1, 3, 4, 5, 6]:

1.
$$\delta^t_M(x) = \begin{cases} 0, & x \in [-1, t) \cap x \in (t, 1], \\ M, & x = t,\ M > 0, \end{cases}$$

$$E^r_{n,\alpha}(\delta^t_1) = \sqrt{1-t^2}\,\frac{\log n}{\alpha n}, \qquad E^r_{n,\alpha}(\delta^1_1) = \frac{2}{\alpha}\left(\frac{\log n}{n}\right)^2 + O\left(\frac{\log n}{n^2}\right);$$

2.
$$\operatorname{sign} x = \begin{cases} -1, & x \in [-1, 0), \\ 0, & x = 0, \\ 1, & x \in (0, 1], \end{cases} \qquad E^r_{n,\alpha}(\operatorname{sign} x) = c(\alpha)\frac{\log n}{n};$$

3.
$$\tau(x) = \begin{cases} 0, & x \in [-1, 0) \cap x \in (0, 1], \\ [-1, 1], & x = 0, \end{cases}$$

$$E^r_{n,\alpha}(\tau) = c(\alpha)\frac{\log n}{n}.$$

To illustrate the advantages which Mathematica9 provides, in the following we describe WEB based moduli for numerical calculation of $E^r_{n,\alpha}(f)$ and corresponding polynomial of the best Hausdorff approximation for the function $\delta^t_M(x)$.

These moduli are written via the programming environment Mathematica9.

One modification of Remez' algorithm for numerical finding of the polynomial of best Hausdorff approximation for the bounded functions is given also in [2].

For the practical importance of the functions described above, see [7].

2. Numerical examples. Let us limit ourselves to calculating the value of the best Hausdorff approximation (with parameter α) of the following two functions

$$M\delta(x) = \delta^0_M(x),\ \ \chi(x) = \delta^1_1(x),$$

by algebraic polynomial of degree less or equal than n.

Following [6] let us remember that if the values of $E^r_{n,\alpha}(M\delta)$ and $E^r_{n,\alpha}(\chi)$ are determined then the corresponding n-order algebraic polynomials P and Q of their best Hausdorff approximation are

$$\begin{aligned} P(x) &= (-1)^{[\frac{n}{2}]} T_{[\frac{n}{2}]}\left(\frac{2x^2 - 1 - \alpha^2(E^r_{n,\alpha}(M\delta)^2}{1 - \alpha^2(E^r_{n,\alpha}(M\delta)^2}\right), \\ Q(x) &= E^r_{n,\alpha}(\chi) T_n\left(\frac{2x + \alpha E^r_{n,\alpha}(\chi)}{2 - \alpha E^r_{n,\alpha}(\chi)}\right), \end{aligned} \tag{1}$$

where $T_n(x) = \cos(n \cos^{-1} x)$ is the Chebyshev polynomial of degree n.

Example 1. For determining the best Hausdorff approximation $\varepsilon = E^r_{n,\alpha}(M\delta)$ of the function $M\delta(x)$, when $n = 70$, $M = 1$, $\alpha = 1$, the following nonlinear equation is used

$$2 - \frac{2}{\varepsilon} + \frac{(1-\varepsilon)^{35}}{(1+\varepsilon)^{35}} + \frac{(1+\varepsilon)^{35}}{(1-\varepsilon)^{35}} = 0. \tag{2}$$

As a result of the execution of the module

(3)

```
n = Input ["Give the degree of the algebraic polynomial - n"]
(*70*)
Print ["The degree of the algebraic polynomial - n = ", n];
k=IntegerPart[n/2];
Print["k=",k];
α = Input ["Give the value of the parameter - α"]; (*1*)
Print ["Parameter α = ", α];
M=Input["Give the value of the parameter - M"];(*1*)
Print["Parameter M=", M];
Print ["The following nonlinear equation is used for determination
of the best Hausdorff approximation ε of the function Mδ(x):"];
s=((1+α*ε)/(1-α*ε))^k+ ((1-α*ε)/(1+α*ε))^k+2-2M/ε;
Print [s, "=0"];
Print["The roots of this equation are: "];
NSolve[s == 0, ε];
Print[TableForm[%]];
Print["The unique positive root of the equation is the
searched value of ε"];
positiveReals = Solve[Reduce[ s==0, ε>0, ε], ε];
If[Length[positiveReals] ==1,
Print["There exists an unique positive root:"
Print[TableForm [N[positiveReals]]];
```

we find the following 53 roots of the equation (2):

$0.0553754 \pm 2.1269000i$	$-0.0536744 \pm 2.0262100i$	$0.0442731 \pm 1.7134200i$
$-0.0432287 \pm 1.6341100i$	$0.0372905 \pm 1.4092900i$	$-0.0366692 \pm 1.3436000i$
$0.0327358 \pm 1.1729600i$	$-0.0324231 \pm 1.1163900i$	$0.0297577 \pm 0.9813700i$
$-0.0297000 \pm 0.9310590i$	$0.0279031 \pm 0.8206260i$	$-0.0280848 \pm 0.7746440i$
$0.0269334 \pm 0.6818490i$	$-0.0273700 \pm 0.6388010i$	$0.0267467 \pm 0.5590540i$
$-0.0274875 \pm 0.5178740i$	$0.0273517 \pm 0.4480330i$	$-0.0284944 \pm 0.4078650i$
$0.0288819 \pm 0.3457240i$	$-0.0306088 \pm 0.3058660i$	$0.0316689 \pm 0.2498790i$
$-0.0343370 \pm 0.2098150i$	$0.0364450 \pm 0.1590680i$	$-0.0407757 \pm 0.1188680i$
$0.0445240 \pm 0.0739267i$	$-0.0503345 \pm 0.0363512i$	**0.0514726**

The unique positive root $E^r_{70,1} = 0.0514726$ is the searched best Hausdorff approximation of degree 70. It follows from (1) that the polynomial

$$(-1)^{35} T_{35} \left(\frac{2x^2 - 1 - (E^r_{70,1}(\delta))^2}{1 - (E^r_{70,1}(\delta))^2} \right)$$

is the algebraic polynomial of best Hausdorff approximation of degree 70 of the function $M\delta(x), M = 1$. Its graph is drawn on Fig. 1 by the module:

```
     n= Input["Give the degree of the algebraic polynomial - n"];(*70*)
     Print["The degree of the algebraic polynomial -  n =",n];
     k=IntegerPart[n/2];
     Print["k = ", k];
     M=Input["Give the value of the parameter - M"];(*1*)
     Print["Parameter M =", M];
     α=Input["Give the value of the parameter - α"];(*1*)
     Print["Parameter α =", α];
(4)  ε=Input["Give the value of the best Hausdorff
     approximation - ε"];(*0.0514726*)
     Print["The value of the best Hausdorff approximation - ε =", ε];
     fk[x_]=0.5((x+(x^2-1)^0.5)^k+ (x-(x^2-1)^0.5)^k);
     P[x_]=(-1)^,k*ε* fk[(2x^2-1-α^2*ε^2)/ (1-α^2*ε^2)];
     Print["Graph of the polynomial of the best Hausdorff
     approximation of Mδ(x)"];
     Show[Plot[p[x],ε,-ε,x,-1,1,PlotRange],
     Plot[1,x,-α*ε, α*ε], PlotRange→-ε,1]
```

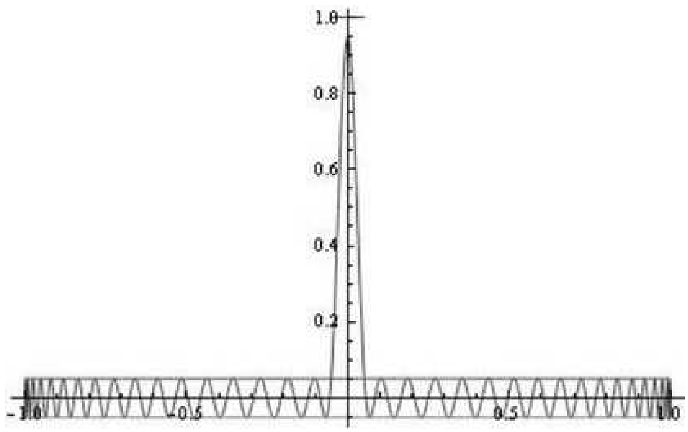

Fig. 1

Example 2. For finding the best Hausdorff approximation $\varepsilon = E^r_{n,\alpha}(\chi)$ of the function $\chi(x)$, when $n = 19, \alpha = 1$, the following nonlinear equation is used

$$2 + \frac{(\sqrt{2}+\sqrt{\alpha\varepsilon})^n}{(\sqrt{2}+\sqrt{\alpha\varepsilon})^n} + \frac{(\sqrt{2}+\sqrt{\alpha\varepsilon})^n}{(\sqrt{2}+\sqrt{\alpha\varepsilon})^n} - \frac{2}{\varepsilon} = 0. \tag{5}$$

A module similar to (3) is used to produce 19 roots of the equation (5):

$-71.774400 \pm 3.880970i$	$-16.958200 \pm 0.993551i$	$-6.824430 \pm 0.459150i$
$-3.299530 \pm 0.272349i$	$-1.694460 \pm 0.185863i$	$-0.854354 \pm 0.138188i$
$-0.386328 \pm 0.107106i$	$-0.129520 \pm 0.080545i$	$-0.007888 \pm 0.046964i$
0.0257161		

The unique positive root $E^r_{19,1}(\chi) = 0.0257161$ of the equation (5) is the searched best approximation and the polynomial of the best Hausdorff approximation of degree 19 is

$$E^r_{19,1}(\chi) T_{19}\left(\frac{2x + E^r_{19,1}(\chi)}{2 - E^r_{19,1}(\chi)}\right). \tag{6}$$

The graph of the polynomial (6) is shown on Fig. 2 using module similar to (4):

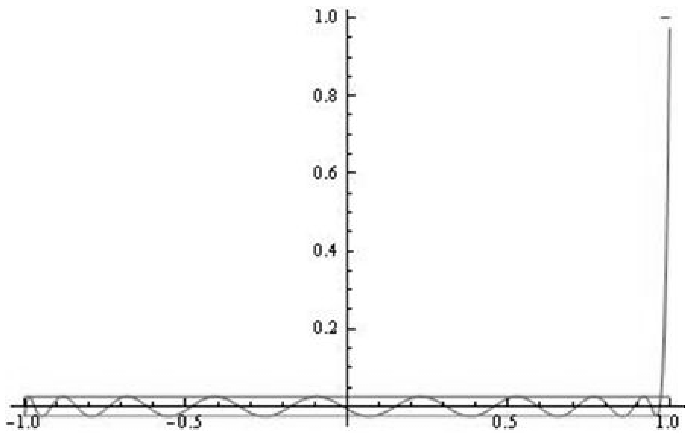

Fig. 2

Remark. The proposed moduli in this paper can be developed to include another animation tools from MATHEMATICA package. For instance, the operator (see [8])

```
Manipulate[Plot[ε*ChebyshevT[n,2/(1+Cos[ε])* Cos[x]+(1-Cos[ε])
/(1+Cos[ε])], {x, -Pi, Pi},PlotRange → Full],
{ε, 0.04, 0.99, Appearance → "Open"},{n,4,160, Appearance → "Open"}]
```

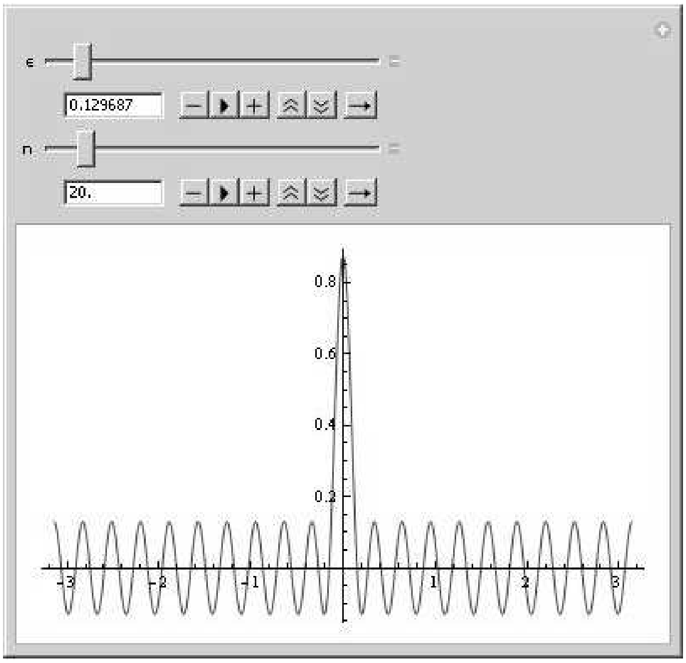

Fig. 3

The approximation by trigonometric polynomial of degree 40 and Hausdorff approximation $\varepsilon = 0.129539$ is shown on Fig. 3 (ε is computed in advance by the help of the module (3)).

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