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A METHOD FOR CLASSIFICATION OF DOUBLY RESOLVABLE DESIGNS AND ITS APPLICATION*

Stela Zhelezova

Abstract. The resolvability of combinatorial designs is intensively investigated because of its applications. This research focuses on resolvable designs with an additional property—they have resolutions which are mutually orthogonal. Such designs are called doubly resolvable. Their specific properties can be used in statistical and cryptographic applications. Therefore the classification of doubly resolvable designs and their sets of mutually orthogonal resolutions might be very important. We develop a method for classification of doubly resolvable designs. Using this method and extending it with some theoretical restrictions we succeed in obtaining a classification of doubly resolvable designs with small parameters. Also we classify 1-parallelisms and 2-parallelisms of $PG(5,2)$ with automorphisms of order 31 and find the first known transitive 2-parallelisms among them. The content of the paper comprises the essentials of the author's Ph.D. thesis.

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Key words: classification; resolvable design; orthogonal resolution; t-parallelism.

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1. Doubly resolvable designs—state of the art. Combinatorial design theory deals with the problems of existence, enumeration and classification of designs and also with the study of their properties and the connection between designs and other combinatorial structures. Kirkman (1847) [48] and Steiner (1853) [75] set up the foundations of combinatorial theory. The main contributions in this area are presented in the monographs of Assmus and Key [2], [3], Beth, Jungnickel and Lenz [5], Cameron and van Lint [11], Hall [35], Hughes and Piper [36], Street A. and Street D. [78], Tonchev [81].

There are many new results in the field of research of combinatorial designs. The Handbook of Combinatorial Designs [14] has gone through several editions because of the dynamic progress in the open cases. Several Bulgarian mathematicians have been working in this field in the last thirty years. Among them are Kapralov [37], [38], Landjev [56], Tonchev [79], [82] and Topalova [83].

Let $V = \{P_i\}_{i=1}^{v}$ be a finite set of *points*, and $\mathcal{B} = \{B_j\}_{j=1}^{b}$ a finite collection of k-element subsets of V, called *blocks*. We say that $D = (V, \mathcal{B})$ is a *design* with parameters 2-(v,k,λ), if any 2-element subset of V is contained in exactly λ blocks of $\mathcal{B}$.

Two designs are *isomorphic* if there exists a one-to-one correspondence between the point set and block collection of the first design and respectively, the point set and block collection of the second design, and if this one-to-one correspondence does not change the incidence.

An *automorphism* of the design is a permutation of the points that transforms the blocks of the design to blocks of the same design.

One of the most important properties of a design is its resolvability. The design is *resolvable* if it has at least one resolution.

A *resolution* is a partition of the blocks into subsets called *parallel classes* such that each point is in exactly one block of each parallel class. A parallel class contains $q = v/k$ blocks and a resolution $\mathcal{R}$ consists of $r = (b * k/v)$ parallel classes, $\mathcal{R} = \mathcal{R}_i, \dots, \mathcal{R}_r$, where r is the number of the blocks containing a definite point.

Two resolutions are isomorphic if there exists an automorphism of the design transforming each parallel class of the first resolution into a parallel class of the second one.

Much work has already been done on the existence or classification of resolvable 2-(v, k, λ) designs with definite parameters, see for instance [17], [42], [43], [45], [63], [64], [65]. A very good recent survey of the different approaches for constructing and classifying design resolutions is contained in [46]. It can be seen from this survey that the most popular construction approach is to generate not the resolution itself, but the corresponding equidistant code. There is a one-to-one correspondence [72] between the resolutions of 2-(qk, k, λ) designs and the $(r, qk, r-\lambda)_q$ equidistant codes, $q > 1$. An equidistant $(r, v, d)_q$ code is a set of v words of length r over an alphabet with q elements, such that the distance between any two distinct codewords is exactly d. The construction of the design resolution is usually done point by point (so in terms of code—word by word) in lexicographic order.

The classification of designs and resolutions is often obtained having in mind some additional properties such as predefined automorphism groups [12], [22], [37], [39], [40], [56], [57], [67], [68], [70], [71], [76], [80], [83], [84], [85], [86], [87], [88]. This is because the problem of making a complete classification of nonisomorphic combinatorial structures requires much more computational time with the growth of parameters.

A parallel class T is *orthogonal* to the resolution $\mathcal{R}$ if $|T \cap \mathcal{R}_i| \leq 1$, $1 \leq i \leq r$. Let both $\mathcal{R} = \mathcal{R}_i, \ldots, \mathcal{R}_r$ and $T = T_i, \ldots, T_r$ be resolutions of one and the same design. These two resolutions are *orthogonal* if $|\mathcal{R}_i \cap T_j| \leq 1$, $1 \leq i, j \leq r$. We call each resolution from an orthogonal pair a *ROR*—a resolution which has an orthogonal one.

When a design has at least two orthogonal resolutions, it is *doubly resolvable* (*DRD*). We denote a set of m mutually orthogonal resolutions by m-MOR and sets of m mutually orthogonal resolutions by m-$MORs$. Two m-$MORs$ are isomorphic if there is an automorphism of the design transforming the first set to the second one. The m-MOR is maximal if no more resolutions can be added to it. We call two m-$MORs$ component equivalent if each resolution of the first m-MOR is isomorphic to some of the resolutions of the second one.

There is a close relation between a pair of orthogonal resolutions and a Kirkman square [1]. A Kirkman square, $KS_k(v;\mu,\lambda)$, with block size k, v points, latinicity μ and index λ is an $r \times r$ array ($r = \lambda(v-1)/\mu(k-1)$) defined on a set V, $|V| = v$ such that: every point of V is contained in precisely μ cells of each row and column; each cell of the array is either empty or contains a k-element subset of V; the collection of blocks obtained from the non-empty cells of the array is a 2-(v, k, λ) design. For μ=1, the existence of a $KS_k(v; \mu, \lambda)$ is equivalent to the existence of a doubly resolvable 2-(v, k, λ) design. In this case the size of the square array r is equivalent to the number of parallel classes of the doubly resolvable design and any two orthogonal resolutions determine a Kirkman square and vice versa.

The main object of the thesis are 2-(v, k, λ) doubly resolvable designs and their sets of orthogonal resolutions. The classification of these combinatorial structures is especially important in view of development of applications in statistics and cryptography. The results obtained by now in the field of $DRDs$ mainly deal with the existence problem—establishing existence or nonexistence of $DRDs$ with certain parameters and setting lower bounds on m for the m-$MORs$.

The papers [1], 2008 and [53], 2009 begin with surveys which give us an idea of the activity of the investigations on $DRDs$. Paper [53] is by Lamken. Her name is the most common in the study of double resolvability. She points out open questions and results regarding the existence of $DRDs$ with certain parameters (some of them are in [15], [16], [21], [24], [49], [50], [51], [52], [54], [55], [96]). There are no classification results among them.

The *starter–adder* method [15], [16], [21], [24], [49], [50], [54], [55], [69], [96] is the most often and very successfully used one and many serious results have been obtained in this field. Another approach that has been used by some authors is to apply orthogonality tests to the resolutions of the classified designs with definite parameters and sometimes additional properties (automorphisms, etc.), see for instance, [12], [47], [76], [80]. But we do not know any method for complete classification of $DRDs$ with certain parameters before the one presented in the thesis.

Goals and problems. The ultimate goal of the thesis is to obtain a complete classification of doubly resolvable designs and their sets of mutually orthogonal resolutions.

To achieve our goal we work on the following problems:

- Development of a method for classification of doubly resolvable designs and its software implementation;
- Classification of doubly resolvable designs with small parameters and the corresponding sets of mutually orthogonal resolutions;
- Application of the constructions to other combinatorial structures related to designs.

The principal aim of the thesis is to find solutions for particular classification problems and therefore it is important how the used algorithms work for specific parameters and if they are applicable to other problems.

2. A method for classification of doubly resolvable designs.

The major part of the thesis is focused on finding an appropriate method for classification of doubly resolvable designs. This problem can be approached in different ways. You may try to construct directly a pair of orthogonal resolutions of designs with certain parameters, but this means that you construct a structure significantly bigger than the resolution itself, too slow for the backtrack search. This is why we look for other solutions of this problem. The number of nonisomorphic resolutions which have an orthogonal one (*RORs*) is much smaller than the number of all nonisomorphic resolutions of the underlying design. So it turned out that it is suitable to construct objects in the following order:

- construction of the nonisomorphic *RORs*;
- classification of the corresponding *DRDs*;
- classification of the nonisomorphic *m*-*MORs*.

This order is especially effective if we find a way for early prunning of inappropriate partial solutions at the first step. We achieve this by application of an orthogonal resolutions existence (*ORE*) test. This test is a basic element of our method. Two constructions are developed in the corresponding chapter of the thesis to implement this test. Also the complexity of the algorithms is calculated and a comparison is made between them in sense of the time needed to obtain a classification of the considered designs.

2.1. Construction of the nonisomorphic ***RORs*****.** The matrix $A = (a_{ij})_{v\times b}$, where $a_{ij} = 1$ if $P_i \in B_j$ and $a_{ij} = 0$ if $P_i \notin B_j$ $(i = 1, 2, \ldots, v,\ j = 1, 2, \ldots, b)$, is called an *incidence matrix* of a 2-(v,k,λ) design.

It has been proved that if a part of the rows (columns) of the incidence matrix of a 2-(v,k,λ) design is known, the problem of extension of this partial solution by rows (columns) is $\mathcal{NP}$-complete [13], [45]. Its special subcase is the problem of the extension of a partial solution of a resolution of a 2-(v,k,λ) design by rows (columns), for which it is not known if there exists a polynomial time algorithm.

To construct resolutions with respect to specific additional conditions we use backtrack search. Let us discuss an example to illustrate the construction.

Example. We consider one of 426 resolutions of the 2-$(9, 3, 3)$ designs, $v = 9$, $k = 3$, $\lambda = 3$, $b = 36$, $q = 3$ and $r = 12$. Its collection of blocks is partitioned into 12 parallel classes with 3 blocks each. Here are the 36 resolution blocks: $b_1 = \{1, 2, 3\}$; $b_2 = \{4, 5, 6\}$; $b_3 = \{7, 8, 9\}$; $b_4 = \{1, 2, 4\}$; $b_5 = \{3, 5, 7\}$; $b_6 = \{6, 8, 9\}$; $b_7 = \{1, 2, 5\}$; $b_8 = \{3, 4, 8\}$; $b_9 = \{6, 7, 9\}$; $b_{10} = \{1, 3, 9\}$; $b_{11} = \{2, 4, 6\}$; $b_{12} = \{5, 7, 8\}$; $b_{13} = \{1, 3, 7\}$; $b_{14} = \{2, 6, 9\}$; $b_{15} = \{4, 5, 8\}$; $b_{16} = \{1, 5, 9\}$; $b_{17} = \{2, 3, 8\}$; $b_{18} = \{4, 6, 7\}$; $b_{19} = \{1, 7, 8\}$; $b_{20} = \{2, 3, 6\}$; $b_{21} = \{4, 5, 9\}$; $b_{22} = \{1, 4, 9\}$; $b_{23} = \{2, 5, 7\}$; $b_{24} = \{3, 6, 8\}$; $b_{25} = \{1, 4, 8\}$; $b_{26} =$

$\{2,7,9\}$; $b_{27} = \{3,5,6\}$; $b_{28} = \{1,6,8\}$; $b_{29} = \{2,4,7\}$; $b_{30} = \{3,5,9\}$; $b_{31} = \{1,5,6\}$; $b_{32} = \{2,8,9\}$; $b_{33} = \{3,4,7\}$; $b_{34} = \{1,6,7\}$; $b_{35} = \{2,5,8\}$; $b_{36} = \{3,4,9\}$ and the incidence matrix is shown in Fig. 1.a.

The corresponding equidistant code has length $r = 12$, $v = 9$ codewords and Hamming distance $d = r - \lambda = 9$ over an alphabet with $q = 3$ elements. Its codewords are presented in Fig. 1.b.

A *ROR* is a resolution with specific additional properties, therefore we begin the construction of the design resolutions with definite parameters. The collection of the design blocks has to be partitioned into parallel classes for the design to be resolvable. Each parallel class consists of disjoint blocks that cover the point set.

By permuting the points, the parallel classes and the blocks within a parallel class, a resolution can be transformed into a resolution with the following properties:

- the first parallel class is fixed and its blocks are $\{1, 2, \ldots, k\}$, $\{k + 1, k + 2, \ldots, 2k\}$, $\ldots$, $\{(q - 1)k + 1, (q - 1)k + 2, \ldots, qk\}$;
- the rows of the incidence matrix are in increasing lexicographic order;
- the blocks of each parallel class and also the parallel classes are in increasing lexicographic order (Fig. 1.a).

Regarding the codewords matrix of the corresponding equidistant code (Fig. 1.b) this means that the first coordinate is always $(0, 0, \ldots, 0, 1, 1, \ldots, 1, \ldots, q, q, \ldots, q)^t$ and the rows and the columns are in lexicographic order.

Let us consider part of the design points $V_x \subset V$, $v_x < v$ and their incidence with the design blocks. We shall call the structure obtained in this manner a *partial solution* on v_x points. If there are no points in some block of the partial solution, this block is 'empty'. We can ask for resolutions and orthogonal resolutions of such a partial solution.

We use the most popular construction approach ([1], [43], [44], [46]) to classify the design resolutions with given parameters. We generate the corresponding equidistant code word by word by backtrack search. Without loss of generality as other authors (see [41, Section 5.2]) we construct only resolutions in the lexicographic order described above.

Each generated partial solution is lexicographically greater than the previous one. This gives us the opportunity to test the obtained partial solutions for minimality (*a minimality test*) to prune early the equivalent ones among them.

a) incidence matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
1	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
2	1	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
3	1	0	0	0	1	0	0	1	0	1	0	0	1	0	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1
4	0	1	0	1	0	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	1	1	0	0	1	0	0	0	1	0	0	0	1	0	0	1
5	0	1	0	0	1	0	1	0	0	0	0	1	0	0	1	1	0	0	0	0	1	0	1	0	0	0	1	0	0	1	1	0	0	0	1	0
6	0	1	0	0	0	1	0	0	1	0	1	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	1	1	0	0	1	0	0	1	0	0
7	0	0	1	0	1	0	0	0	1	0	0	1	1	0	0	0	0	1	1	0	0	0	1	0	0	1	0	0	1	0	0	0	1	1	0	0
8	0	0	1	0	0	1	0	1	0	0	0	1	0	0	1	0	1	0	1	0	0	0	0	1	1	0	0	1	0	0	0	1	0	0	1	0
9	0	0	1	0	0	1	0	0	1	1	0	0	0	1	0	1	0	0	0	0	1	1	0	0	0	1	0	0	0	1	0	1	0	0	0	1

b) corresponding equidistant $(12, 9, 9)_3$ code

$$
\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\
1 & 0 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 1 & 2 & 2 \\
1 & 1 & 0 & 2 & 2 & 0 & 2 & 1 & 2 & 2 & 0 & 1 \\
1 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 0 & 0 & 0 \\
2 & 1 & 2 & 2 & 0 & 2 & 0 & 1 & 1 & 1 & 2 & 0 \\
2 & 2 & 1 & 2 & 2 & 1 & 0 & 2 & 0 & 0 & 1 & 1 \\
2 & 2 & 0 & 1 & 1 & 0 & 2 & 0 & 1 & 2 & 1 & 2
\end{matrix}
$$

c) 2-MOR

$$
\mathcal{R} = \begin{matrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
10 & 11 & 12 \\
13 & 14 & 15 \\
16 & 17 & 18 \\
19 & 20 & 21 \\
22 & 23 & 24 \\
25 & 26 & 27 \\
28 & 29 & 30 \\
31 & 32 & 33 \\
34 & 35 & 36
\end{matrix}
\qquad
\mathcal{T} = \begin{matrix}
1 & 15 & 9 \\
4 & 27 & 3 \\
7 & 33 & 6 \\
10 & 35 & 18 \\
13 & 32 & 2 \\
16 & 29 & 24 \\
22 & 20 & 12 \\
25 & 14 & 5 \\
28 & 23 & 36 \\
31 & 26 & 8 \\
34 & 17 & 21 \\
19 & 11 & 30
\end{matrix}
$$

Fig. 1. A 2-$(9, 3, 3)$ design

Equivalent codes are obtained by permuting words, coordinates and symbols coordinatewise. We check if there exists a partial solution equivalent to the current one and lexicographically smaller than it. For this purpose the permutations of the points are generated in lexicographic order, starting with $\varphi(i) = i$ for $i = 1, 2, \ldots, v$ then $\varphi(i) = i$ for $i = 1, 2, \ldots, v-2$, $\varphi(v-1) = v$ and $\varphi(v) = v-1$, etc. In each coordinate we apply the permutation of the symbols (for a given permutation of the points) that transforms the corresponding coordinate into the lexicographically smallest one. Then the coordinates are ordered lexicographically.

- If the obtained partial solution is lexicographically smaller than the current one, we skip it as already considered and we generate the next solution.
- If the obtained partial solution is lexicographically greater than the current one, we look for the smallest point u, $\varphi(u) = v_u$, such that the partial solution on the points $\varphi(1)$, $\varphi(2)$, $\ldots$, $\varphi(u)$ is lexicographically greater than the partial solution on the points $1, 2, \ldots, u$. The next permutation φ is generated such that $\varphi(u) > v_u$.
- If the obtained partial solution is equal to the current one (i.e. the permutation φ is an automorphism of the resolution), we look for the greatest point u such that $\varphi(i) = i$ for each $i \leq u$. The next permutation φ is generated such that $\varphi(u) > u$.

The next codeword is generated when the lexicographically greatest permutation is checked and a partial solution equivalent to the current one and lexicographically smaller than it is not found.

The details of such a technique are described, for instance, in [9], [10], [41], [45, Section.7.1.2], [83]. Kaski and Östergård in [41] and [45, Section.7.1.2] call it canonicity test.

We apply a minimality test after each added codeword if the number of the words (number of the points of a partial solution) is at most v_e. Usually we set $v_e = 9$ or $v_e = 10$. Greater values of v_e require much more computational time. We do not apply the minimality test if the words are more than v_e and use it only on complete solutions for *ROR*s (v words).

A design must have at least two orthogonal resolutions to be doubly resolvable. The check if a particular resolution has an orthogonal one is done by finding a new partition of the design blocks such that each constructed orthogonal parallel class has at most one common block with each of the initial resolution's parallel classes. This problem is a particular case of the problem of whether a design is resolvable or not. No polynomial-time algorithm is known for the general case of the resolvability problem. We know of no other authors applying an orthogonal resolution existence test on partial solutions. The two constructions we use are based on backtrack search.

Let each of the blocks in our example have a consecutive number from the first (1) to the last (36). The considered resolution (Fig. 1.a) is denoted in Fig. 1.c by $\mathcal{R}$. It is written by the blocks within each parallel class there—each parallel class is on a particular row. $\mathcal{T}$ is a resolution orthogonal to $\mathcal{R}$. It can be seen, for instance, that the blocks 1, 2 and 3 from the first parallel class of $\mathcal{R}$ are in different parallel classes of $\mathcal{T}$—namely they are in the first, second and fifth parallel class of $\mathcal{T}$.

Not all resolutions have an orthogonal one, therefore the application of the *ORE* test on partial solutions as early as possible can significantly reduce the search. We experimented on the resolutions' partial solutions with different parameters to establish a connection between the number of the partial solution points and the result of the *ORE* test.

If the whole resolution is a *ROR*, a partial solution (on only v_x points) obviously has an orthogonal partial solution too, so the test result is positive. If the resolution has no orthogonal one, a partial solution may have an orthogonal one. Depending on the number v_x of the points of partial solutions, some of the blocks may contain fewer than k points or even no points at all, and therefore for these blocks there are many more ways to participate in different orthogonal parallel classes. The *ORE* test result may be positive in this case although the whole resolution is not a *ROR*. The greater the number v_x of partial solution points, the smaller is the time needed for the *ORE* test to reject the solution if it does not lead to a *ROR*.

Let us consider a resolution of the 2-$(9,3,3)$ design, which is not a *ROR*. Its blocks are: $b_1 = \{1,2,3\}$, $b_2 = \{4,5,6\}$, $b_3 = \{7,8,9\}$, $b_4 = \{1,2,3\}$, $b_5 = \{4,5,6\}$, $b_6 = \{7,8,9\}$, $b_7 = \{1,2,3\}$, $b_8 = \{4,5,6\}$, $b_9 = \{7,8,9\}$, $b_{10} = \{1,4,7\}$, $b_{11} = \{2,5,8\}$, $b_{12} = \{3,6,9\}$, $b_{13} = \{1,4,7\}$, $b_{14} = \{2,5,8\}$, $b_{15} = \{3,6,9\}$, $b_{16} = \{1,4,7\}$, $b_{17} = \{2,5,9\}$, $b_{18} = \{3,6,8\}$, $b_{19} = \{1,6,8\}$, $b_{20} = \{2,4,9\}$, $b_{21} = \{3,5,7\}$, $b_{22} = \{1,6,8\}$, $b_{23} = \{2,4,9\}$, $b_{24} = \{3,5,7\}$, $b_{25} = \{1,6,9\}$, $b_{26} = \{2,4,8\}$, $b_{27} = \{3,5,7\}$, $b_{28} = \{1,5,8\}$, $b_{29} = \{2,6,7\}$, $b_{30} = \{3,4,9\}$, $b_{31} = \{1,5,9\}$, $b_{32} = \{2,6,7\}$, $b_{33} = \{3,4,8\}$, $b_{34} = \{1,5,9\}$, $b_{35} = \{2,6,7\}$, $b_{36} = \{3,4,8\}$.

We consider its partial solution on 7 points. The incidence matrix of this partial solution and its orthogonal one are in Fig. 2.a and Fig. 2.b, respectively. For this partial solution the *ORE* test result will be positive although the whole resolution is not a ROR. Adding a point to the considered partial solution we obtain a partial solution on 8 points. Its incidence matrix is in Fig. 2.c. When we try to find an orthogonal partial solution we will see that such a solution does not exist. Therefore the ORE test result is negative and this partial solution is rejected and we generate the next partial solution on 8 points.

Contrary to the minimality test, the ORE test works faster if the number of points is greater, and we do not apply it to fewer than v_o points. The efficiency of the method is very sensitive to the choice of v_e and v_o. Therefore for particular parameters it is necessary to investigate the inclusion points of the two tests experimentally at first. For the parameters we usually consider that $v/2 \leq v_o \leq 2v/3$ leads to good results for the speed of the classification of RORs.

We have implemented two algorithms for the ORE test. They both use backtrack search to partition the blocks of the current partial solution into its orthogonal one. The search stops if one such partition is constructed, or if all possibilities have been tested and no partition can be found. In both constructions we first have to change the form of the considered partial solution from the equidistant code matrix to the incidence matrix of the corresponding design. The first construction tries to obtain an orthogonal solution block by block (BB) and the second one class by class (CC).

By the BB construction we sort the blocks of the design in lexicographic order. The first point is in the first r blocks. Thus without loss of generality we assume that for $i = 1, 2, \ldots, r$ the ith block is in the ith parallel class of the orthogonal resolution. We start adding the missing blocks to the first class of the orthogonal resolution, then to the second, ..., and finally to the rth. Since an orthogonal parallel class should contain all points, at each step we try to add only blocks containing the first missing point in the class. To add a block to the current class we check the following conditions:

- the block is disjoint to all previously added blocks;
- the initial resolution class of the block is different from the classes of the previously added blocks.

If we apply this algorithm to the resolution $\mathcal{R}$ from the example in Figure 1, the result will be as follows. We begin to construct the first orthogonal parallel class. The block $b_1 = \{1, 2, 3\}$ is there. The first missing point for this class is 4. The blocks are sorted in the following lexicographic order (written by their initial numbers): 1, 4, 7, 13, 10, 25, 22, 31, 16, 34, 28, 19, 20, 17, 11, 29, 23, 35, 14, 26, 32, 33, 8, 36, 27, 5, 30, 24, 2, 15, 21, 18, 12, 9, 6, 3. We check the first block

a) partial solution on 7 points

	1 2 3	4 5 6	7 8 9	101112	131415	161718	192021	222324	252627	282930	313233	343536
1	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0
2	1 0 0	1 0 0	1 0 0	0 1 0	0 1 0	0 1 0	0 1 0	0 1 0	0 1 0	0 1 0	0 1 0	0 1 0
3	1 0 0	1 0 0	1 0 0	0 0 1	0 0 1	0 0 1	0 0 1	0 0 1	0 0 1	0 0 1	0 0 1	0 0 1
4	0 1 0	0 1 0	0 1 0	1 0 0	1 0 0	1 0 0	0 1 0	0 1 0	0 1 0	0 0 1	0 0 1	0 0 1
5	0 1 0	0 1 0	0 1 0	0 1 0	0 1 0	0 1 0	0 0 1	0 0 1	0 0 1	1 0 0	1 0 0	1 0 0
6	0 1 0	0 1 0	0 1 0	0 0 1	0 0 1	0 0 1	1 0 0	1 0 0	1 0 0	0 1 0	0 1 0	0 1 0
7	0 0 1	0 0 1	0 0 1	1 0 0	1 0 0	1 0 0	0 0 1	0 0 1	0 0 1	0 1 0	0 1 0	0 1 0

b)2-MOR on 7 points

$$\mathcal{R}_{v_7} = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \\ 19 & 20 & 21 \\ 22 & 23 & 24 \\ 25 & 26 & 27 \\ 28 & 29 & 30 \\ 31 & 32 & 33 \\ 34 & 35 & 36 \end{array} \qquad \mathcal{T}_{v_7} = \begin{array}{ccc} 1 & 5 & 9 \\ 2 & 7 & 6 \\ 3 & 4 & 8 \\ 10 & 14 & 18 \\ 11 & 15 & 16 \\ 12 & 13 & 17 \\ 19 & 23 & 27 \\ 20 & 24 & 25 \\ 21 & 22 & 26 \\ 28 & 32 & 36 \\ 29 & 33 & 34 \\ 30 & 31 & 35 \end{array}$$

c) partial solution on 8 points

	1 2 3	4 5 6	7 8 9	101112	131415	161718	192021	222324	252627	282930	313233	343536
1	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0	1 0 0
2	1 0 0	1 0 0	1 0 0	0 1 0	0 1 0	0 1 0	0 1 0	0 1 0	0 1 0	0 1 0	0 1 0	0 1 0
3	1 0 0	1 0 0	1 0 0	0 0 1	0 0 1	0 0 1	0 0 1	0 0 1	0 0 1	0 0 1	0 0 1	0 0 1
4	0 1 0	0 1 0	0 1 0	1 0 0	1 0 0	1 0 0	0 1 0	0 1 0	0 1 0	0 0 1	0 0 1	0 0 1
5	0 1 0	0 1 0	0 1 0	0 1 0	0 1 0	0 1 0	0 0 1	0 0 1	0 0 1	1 0 0	1 0 0	1 0 0
6	0 1 0	0 1 0	0 1 0	0 0 1	0 0 1	0 0 1	1 0 0	1 0 0	1 0 0	0 1 0	0 1 0	0 1 0
7	0 0 1	0 0 1	0 0 1	1 0 0	1 0 0	1 0 0	0 0 1	0 0 1	0 0 1	0 1 0	0 1 0	0 1 0
8	0 0 1	0 0 1	0 0 1	0 1 0	0 1 0	0 0 1	1 0 0	1 0 0	0 1 0	1 0 0	0 0 1	0 0 1

Fig. 2. Partial solutions for a resolution of the 2-$(9, 3, 3)$ design which is not a *ROR*

in this order starting with the point 4, the block b_2. It is not suitable because it is from the initial resolution's first parallel class—from the same class as b_1 which is already in the constructed orthogonal class. The next suitable block is $b_{15} = \{4, 5, 8\}$. It corresponds to the required conditions, so we add it to the first orthogonal class. The first missing point this time is 6. The first block in our lexicographic order starting with this point is b_9. It fits our conditions, so the first orthogonal parallel class is completed. It consists of the blocks b_1, b_{15} and b_9. We continue to construct the second orthogonal parallel class in the same way until the resolution $\mathcal{T}$ (Fig. 1.c) is complete.

Not all the blocks are checked in the *BB* construction and this is the reason it works relatively fast. We stop adding blocks to an orthogonal parallel class when all points of the partial solution are covered although fewer than q blocks are added.

In the *CC* construction at the beginning we generate, for each of the first r blocks, all possibilities for a parallel class containing this block and orthogonal to the initial resolution. Each orthogonal class consists of q blocks from different parallel classes of the initial resolution. Therefore we write each possibility for an orthogonal parallel class as a vector a of length r—$(a_1, a_2, \ldots, a_r)$, where $a_i = 1, 2, \ldots, q$ is the number of a block within the ith parallel class ($i = 1, 2, \ldots, r$) of the initial resolution, or $a_i = 0$ if the orthogonal class has no block from the ith parallel class of the initial resolution. Equality of two elements a_i and a_j, $1 \le i, j \le r$ is possible, because blocks can be in the same position in different parallel classes. Each vector is with q nonzero and $r - q$ zero positions. We choose as the first nonzero element of the vector each of the blocks from the first to the $r - q + 1$-th initial parallel classes and we use backtrack search on the next ones. From each class we can choose 1 or 0 blocks. Each chosen block has to be disjoint with previously added ones. When all points are covered the vector corresponding to an orthogonal parallel class is done. The *ORE* test returns a negative result if some of the design blocks do not participate in any vector.

For the resolution $\mathcal{T}$ in Fig. 1.c one such vector is (1, 0, 3, 0, 3, 0, 0, 0, 0, 0, 0, 0). This means that the orthogonal parallel class consists of the block 1 from the first parallel class of the initial resolution $\mathcal{R}$, the block 3 from the 3rd one and the block 3 from the 5th one (the blocks b_1, b_9 and b_{15} of the initial resolution). All vectors for the resolution $\mathcal{R}$ obtained by the *CC* construction can be seen in Fig. 3.

The next step in the *CC* construction is to choose r among the obtained parallel class vectors to form an orthogonal resolution. This means we try to construct a Kirkman square for the given parameters. We have implemented the

1	0	3	0	3	0	0	0	0	0	0	0
2	0	0	0	1	0	0	0	0	0	2	0
3	1	0	0	0	0	0	0	3	0	0	0
0	2	0	0	2	0	0	0	1	0	0	0
0	3	1	0	0	0	0	0	0	0	3	0
0	0	2	0	0	0	0	0	2	0	1	0
0	0	0	1	0	3	0	0	0	0	0	2
0	0	0	2	0	0	1	0	0	3	0	0
0	0	0	3	0	0	2	1	0	0	0	0
0	0	0	0	0	1	0	3	0	2	0	0
0	0	0	0	0	2	3	0	0	0	0	1
0	0	0	0	0	0	0	2	0	1	0	3

Fig. 3. The vectors for a parallel class orthogonal to $\mathcal{R}$

following algorithm. First we sort the design blocks by the number of the vectors they participate in. Then we try to construct an orthogonal resolution using backtrack search on the sorted vectors.

The Kirkman square corresponding to the orthogonal resolutions $\mathcal{R}$ and $\mathcal{T}$ from our example is presented in Fig. 4. Both resolutions can be obtained by taking as parallel classes columns for $\mathcal{R}$ and rows for $\mathcal{T}$ respectively.

$1,2,3$		$6,7,9$		$4,5,8$							
$4,5,6$				$1,3,7$						$2,8,9$	
$7,8,9$	$1,2,4$							$3,5,6$			
	$3,5,7$			$2,6,9$				$1,4,8$			
	$6,8,9$	$1,2,5$								$3,4,7$	
		$3,4,8$						$2,7,9$		$1,5,6$	
			$1,3,9$		$4,6,7$						$2,5,8$
			$2,4,6$			$1,7,8$			$3,5,9$		
			$5,7,8$			$2,3,6$	$1,4,9$				
					$1,5,9$		$3,6,8$		$2,4,7$		
					$2,3,8$	$4,5,9$					$1,6,7$
							$2,5,7$		$1,6,8$		$3,4,9$

Fig. 4. $KS_3(9;1,3)$

When we apply the ORE test by the CC construction on partial solutions with an empty block, instead of the number of this block within the corresponding class we write 0. In this case there are more than $r-q$ zero positions in the vector of an orthogonal parallel class, but not more than the number of all empty blocks in the initial partial solution. Therefore the zero in some position has already two meanings: a block is not chosen from this parallel class of the initial resolution or there is an empty block in this class. If we consider a vector with $r-q$ zero symbols and all points are covered we cannot choose a zero value for the next vector position if it corresponds to a parallel class without an empty block.

Our implementation of both constructions is in C++ and the asymptotic time complexity of the algorithm is discussed in the thesis.

We choose $q-1$ blocks for each orthogonal parallel class in the BB construction. Each one is chosen among no more than r blocks beginning with the first missing point in the class. The parallel classes are r and therefore in the worst case the time complexity is $\mathcal{O}(r^{(q-1)r})$ and in the best case it is $\Omega((q-1)r)$.

The number of elementary operations for the generation of the vectors in the *CC* construction is $\mathcal{O}((q+1)^r)$ because there are $q+1$ possibilities for each vector position. All possibilities for an orthogonal parallel class are at most $\binom{r}{q}q^q$ because $r-q$ zeros in the vector are fixed in $\binom{r}{q}$ ways and the nonzero positions are q, each being chosen among q blocks of the corresponding initial resolution parallel class. Therefore in the worst case the time complexity is $\mathcal{O}(\binom{r}{q}^r q^{rq}) = O((rq)^{rq})$ and in the best case it is $\Omega((q+1)^r)$.

The asymptotic time complexity of the algorithm for the *CC* construction is greater but it turns out that for the problems we consider both constructions are appropriate.

2.2. Classification of doubly resolvable designs. As a result of the application of the *BB* and *CC* constructions mentioned above we obtain all nonisomorphic design resolutions (with certain parameters) which have an orthogonal one (*ROR*s). In general a doubly resolvable design may have several nonisomorphic *RORs*. Therefore at this step of our method we consider the obtained *RORs* as designs and take away the isomorphic ones by a design isomorphism test. (There are many known implementations of such a test. We use the software by Topalova [83].) As a result only nonisomorphic *DRD*s remain.

2.3. Classification of the nonisomorphic sets of mutually orthogonal resolutions. We have all doubly resolvable designs with certain parameters and we start with a *DRD*. At first we find the automorphism group of the design. Next we construct its resolutions block by block. For each resolution $\mathcal{R}_1$ we check if it is isomorphic to a lexicographically smaller one. If not, we try to construct another resolution $\mathcal{R}_2$ which is lexicographically greater and orthogonal to $\mathcal{R}_1$. Then we repeat the same for $\mathcal{R}_3$, orthogonal to both $\mathcal{R}_1$ and $\mathcal{R}_2$, etc. We apply point automorphisms to the resolutions of the i-MOR obtained at this step after having constructed the resolution $\mathcal{R}_i$, $i > 1$. We check if some automorphism and some suitable permutation of the numbers of one and the same block maps some resolutions of the i-MOR to a resolution lexicographically smaller than $\mathcal{R}_1$. If it is mapped exactly to $\mathcal{R}_1$ we check in the same way if some of the remaining resolutions of the i-MOR is mapped to a value lexicographically smaller than $\mathcal{R}_2$ or exactly to $\mathcal{R}_2$, etc. In this way we obtain nonisomorphic m-MORs.

3. On the application of the method for classification of doubly resolvable designs. Using the method described above we manage to obtain the first classification of doubly resolvable designs with small parameters. We also apply a modification of the method to resolutions of designs related to $PG(5,2)$ and thus we classify 1-parallelisms and 2-parallelisms of $PG(5,2)$ with automorphisms of order 31.

3.1. Investigations on the structure of orthogonal resolutions. To extend the parameter range of our method we study the structure of orthogonal resolutions in the corresponding chapter of the thesis. We have tried to find and use parameter-specific restrictions which double resolvability imposes on the intersection possibilities of the parallel classes. The parallel classes intersection matrix ($PCIM$) has been defined and used in [42], [63] and [64] for the classification of resolvable designs with several parameter sets.

The restrictions established in the thesis help us to prove the non-existence of DRDs with parameters 2-$(2k, k, k-1)$, 2-$(14, 7, 12)$ and 2-$(18, 9, 16)$. The application of these restrictions to the construction of resolutions and $RORs$ makes the classification of $RORs$ of the 2-$(20, 10, 18)$ designs possible and speeds up the computation for other parameter sets. We also improve the bounds on the number of all resolutions of 2-$(12, 6, 15)$ and 2-$(16, 8, 14)$ designs.

A Steiner triple system of order v ($STS(v)$) is a 2-$(v, 3, 1)$ design. Steiner triple systems are fully classified for $v <= 19$. For the next value $v = 21$ a complete classification is currently out of reach [62], but various classification results on STS(21) with additional properties exist [12], [39], [47], [60], [61], [80], [88]. All these authors also test the obtained $STS(21)$s for resolvability. The problem of the existence of a doubly resolvable Steiner triple system of order 21 is still open with 21 being the smallest value of v for which it is not known if a doubly resolvable $STS(v)$ exists or not.

We consider some specific properties of $KTS(21)$ and set restrictions on partial solutions for a ROR with these parameters. These peculiarities of a

$KTS(21)$ help us split the problem into several cases, so that subsequent computer search can be applied to them.

Lemma 3.1. *Let $D = (V, B)$ be a $KTS(21)$. Consider three of the triples of one parallel class. They define a partial solution on a 9-element subset $V_9 \subset V$. Let t_9 be the number of triples on the points of V_9. Let $V'_{12} = V \setminus V_9$ and t'_{12} be the number of triples on the points of V'_{12}. Then*

a. $3 \leq t_9 \leq 12$;
b. $t'_{12} = 16 - t_9$;
c. $4 \leq t'_{12} \leq 13$.

From this lemma it follows:

Corollary 3.2. *Let $D = (V, B)$ be a $KTS(21)$. Consider four of the triples of one parallel class. They define a partial solution on a 12-element subset $V_{12} \subset V$. Let t_{12} be the number of triples on the points of V_{12}. Then $4 \leq t_{12} \leq 13$.*

We use Lemma 3.1 and Corollary 3.2 to prove the following Theorems.

Theorem 3.3. *Let $D = (V, B)$ be a $KTS(21)$. Then*

1. *There exists a* 9*-element subset $V_9 \subset V$, such that $t_9 = 3$ or $t_9 = 4$ and three of them are nonintersecting.*

2. *V_9 is contained in a* 12*-element subset $V_{12} \subset V$, such that $t_{12} = 11$, $t_{12} = 12$, or $t_{12} = 13$, at least* 4 *of them are nonintersecting.*

Theorem 3.4. *Let $D = (V, B)$ be a doubly resolvable $STS(21)$ with two mutually orthogonal resolutions respectively $\mathcal{R}$ and $\mathcal{T}$. Consider three of the triples of one parallel class R_1 of $\mathcal{R}$. They define a* 9*-element subset V_9 of V. Then there are at most 11 triples on the points of V_9.*

This result coincides with [47], whose authors claim that no $STS(21)$ having a Steiner triple subsystem is doubly resolvable.

Theorem 3.5. *Let $D = (V, B)$ be a $KTS(21)$, $V_9 \subset V_{12}$ and the number of the triples on the points of V_9 and V_{12} be* 4 *and* 13*. This case is equivalent to a case with* 3 *and* 12 *triples respectively on the points of V_9 and V_{12}.*

The application of Theorems 3.3, 3.4 and 3.5 allows us to split the problem into five cases, with respect to the number of triples on partial solutions on 9 and 12 points. For one subcase of them we have checked by computer that it doesn't lead to a doubly resolvable $STS(21)$.

The investigations on the structure of $KTS(21)$ described in the thesis allow us to eliminate solutions which cannot lead to a ROR. Even with the strong restrictions from the Theorems and the ORE test from the 14th point on, this still remains a very hard computational problem. The problem of the existence of a doubly resolvable Steiner triple system of order 21 is still open.

3.2. Classification of doubly resolvable designs with small parameters. The obtained classification results are summarized in tables in the corresponding chapter of the thesis. There are also the parameter range of the method, the first open cases and our results on lower bounds for the number of all design resolutions with some parameters.

The classification time for the results presented in the thesis varies from several minutes to several days. We have not found classification results for $RORs$ and $DRDs$ announced by other autors.

Here we present two of the classification tables in the thesis. The computational results for $RORs$ and $DRDs$ are in Table 1 where "Nr" is the number of nonisomorphic resolutions known by now. The value is presented without any comments if it is taken from [62]. It is followed by $\surd$ if we too have obtained the same number by our program independently, and by $*$ if it is obtained by our program and we do not know a better bound calculated by other authors. In the column "No" the number of the design in the tables of [62] is given.

Table 1. Computational results for $RORs$ and $DRDs$

q	v	k	λ	N_r		ROR	DRD	No
2	6	3	4	1	$\surd$	0	0	43
2	6	3	$4n$	1	$\surd$	1	1	$2 \leq n \leq 13$
2	8	4	3	1	$\surd$	0	0	15
2	8	4	6	4	$\surd$	1	1	101
2	8	4	9	10	$\surd$	1	1	278
2	8	4	12	31	$\surd$	4	4	524
2	8	4	15	82	$\surd$	4	4	819
2	8	4	18	240	$*$	13	13	-
2	8	4	21	650	$*$	16	16	-
2	8	4	24	1803	$*$	44	44	-
2	8	4	27	4763	$*$	70	70	-
2	10	5	8	5	$\surd$	0	0	195
2	10	5	16	27121734		5	5	891
2	10	5	24	$\geq$73534	$*$	6	6	-

2	12	6	5	1 $\surd$	0	0	58
2	12	6	10	545 $\surd$	1	1	319
2	12	6	15	$\geq$128284 *	1	1	743
2	12	6	20	$\geq$546 *	546	546	-
2	14	7	12	1363486	0	0	451
2	16	8	7	5 $\surd$	0	0	130
2	16	8	14	$\geq$1895 *	5	5	618
2	16	8	21	$\geq$5 *	5	5	-
2	18	9	16	≥ 1	0	0	791
2	20	10	9	3 $\surd$	0	0	224
2	20	10	18	≥ 4	3	3	1007
2	24	12	11	130	0	0	346
2	28	14	13	7570	0	0	499
2	32	16	15	$\geq$1	0	0	668
3	9	3	1	1 $\surd$	0	0	2
3	9	3	2	9 $\surd$	0	0	21
3	9	3	3	426 $\surd$	5	3	66
3	9	3	4	149041 $\surd$	83	38	145
3	9	3	5	203047732	$\geq$76992	$\geq$27269	235
3	12	4	3	5 $\surd$	0	0	56
3	27	9	4	68	0	0	90
4	12	3	2	74700	70	20	55
4	16	4	1	1 $\surd$	0	0	5
4	16	4	2	339592	1	1	44
5	15	3	1	7 $\surd$	0	0	14
5	20	4	3	≥ 204			220
5	25	5	1	1	0	0	11
7	28	4	1	7	0	0	32

In Table 2 computational results for *m-MORs* are presented. The first value is the number of maximal m-MORs, and the second is all m-MORs for $m = 2, 3, 4$.

The full classification covers *RORs* and *DRDs* for 9 design parameter sets for which resolutions are not classified. There are only two parameter sets

Table 2. Computational results for m-MOR

q	v	k	λ	ROR	DRD	2-MOR		3-MOR		4-MOR		No
2	6	3	8	1	1	1	1	-	-	-	-	236
2	6	3	12	1	1	0	1	1	1	-	-	596
2	6	3	16	1	1	0	$\geq$15	0	$\geq$485	$\geq$485	$\geq$485	1078
2	8	4	6	1	1	1	1	-	-	-	-	101
2	8	4	9	1	1	0	1	1	1	-	-	278
2	8	4	12	4	4	7	17	0	60	60	60	524
2	10	5	16	5	5	5	5	-	-	-	-	891
2	10	5	24	6	6	2	7	5	5	-	-	-
2	12	6	10	1	1	1	1	-	-	-	-	319
2	12	6	15	1	1	0	1	1	1	-	-	743
2	12	6	20	546	546	691	$\geq$ 701	0	$\geq$223	$\geq$223	$\geq$223	-
2	16	8	14	5	5	5	5	-	-	-	-	618
2	16	8	21	5	5	0	5	5	5	-	-	-
2	20	10	18	3	3	3	3	-	-	-	-	1007
3	9	3	3	5	3	2	7	5	5	-	-	66
3	9	3	4	83	38	351	449	284	285	1	1	145
4	12	3	2	70	20	252	254	1	2	1	1	55
4	16	4	2	1	1	0	1	1	1	-	-	44

for which full classification of the resolutions is known, but we cannot construct the $RORs$ by the method described above, namely 2-$(9,3,5)$ [43] and 2-$(28,4,1)$ [46]. For the former case the expected number of $RORs$ is quite large, while in the latter case we find that there are no $RORs$ by applying the ORE test to the resolutions of 2-$(28,4,1)$ designs.

As was already said in the introduction, many existence results have been obtained by numerous other authors. These results are not included in our classification tables. The files with all $RORs$, $DRDs$ and m-$MORs$ constructed in the thesis can be downloaded from `http://www.moi.math.bas.bg/moiuser/~stela`.

Sets of m mutually orthogonal resolutions (m-$MORs$) of 2-$(v,k,m\lambda)$ designs are also considered in the thesis. The 2-$(v,k,m\lambda)$ designs are called *quasi-multiples* of a 2-(v,k,λ) design.

The existence of quasimultiples of the designs for greater values of λ determine the very fast growth of the number of their $RORs$ and m-$MORs$. This is the reason why we cannot use our method to classify the m-$MORs$ of all doubly resolvable designs classified by us.

We investigate quasimultiples of the designs and derive a dependence of the number of their nonisomorphic *RORs* and m-*MORs* on the number of inequivalent sets of $q-1$ mutually orthogonal Latin squares ($MOLS$) of side m.

Theorem 3.6. *Let D be a 2-(v,k,λ) design and $v = 2k$.*

1. *D is doubly resolvable iff it is resolvable and each set of k points is incident either with no block, or with at least two blocks of the design.*
2. *If D is doubly resolvable and at least one set of k points is in m blocks, and the rest in 0 or more than m blocks, then D has at least one maximal m-MOR, no μ-MORs for $\mu > m$ and no maximal μ-MORs for $\mu < m$.*

Theorem 3.7. *Let $l_{q-1,m}$ be the number of main class inequivalent sets of $q-1$ MOLS of side m, $m \geq q$. Let the 2-$(v,k,m\lambda)$ design D be decomposed into m copies of a resolvable 2-(v,k,λ) design d. If $l_{q-1,m} > 0$, then D is doubly resolvable and has at least $\binom{\frac{r}{m} - 1 + l_{q-1,m}}{\frac{r}{m}}$ m-MORs which are nonisomorphic, but component equivalent.*

The exact values of $l_{q-1,m}$ for $q > 2$ are not known. It is a complicated problem in itself. For many parameters it is known whether there exists a set of $q-1$ MOLS of side m ($l_{q-1,m} > 0$). This can be used to establish existence of m-*MORs*.

The number of main class inequivalent Latin squares of side m is known for many values of m, and thus for $q = 2$ much better bounds can be set using the next corollary, which follows directly from Theorems 3.6 and Theorem 3.7.

Corollary 3.8. *Let l_m be the number of main class inequivalent Latin squares of side m. Let $q = 2$ and $m \geq 2$. Let the 2-$(v,k,m\lambda)$ design D be decomposable into m copies of a resolvable 2-(v,k,λ) design d. Then D is doubly resolvable and has at least $\binom{\frac{r}{m} - 1 + l_m}{\frac{r}{m}}$ nonisomorphic, but component equivalent m-MORs, no maximal i-MORs for $i < m$, and if d is not doubly resolvable, no i-MORs for $i > m$.*

By Corollary 3.9 we can calculate lower bounds on the number of *RORs* for some parameters. If some 2-(v,k,λ) designs have many resolutions, the number of *RORs* of the 2-$(v,k,m\lambda)$ designs consisting of m of their copies grows very fast with the parameters. This is why the classification of m-$MORs$ for such a great number of $RORs$ is very difficult in reasonable time.

Corollary 3.9. *Let N_d be the number of nonisomorphic resolvable 2-(v,k,λ) designs, and N_r the number of their nonisomorphic resolutions. The number of nonisomorphic RORs and m-MORs of 2-$(v,k,m\lambda)$ designs with $m \geq q$ is greater or equal to N_r and $max(N_r/m, N_d)$ respectively.*

3.3. Construction of t-spreads and t-parallelisms in $PG(5,2)$. There is a relation between the resolutions of the designs related to a certain projective space and t-parallelisms of this space.

A *t-spread* in $PG(d,q)$ is a set of t-dimensional subspaces which partition the point set. In particular a 1-spread is a set of lines which partition the point set. It is usually simply called *spread*.

A *partial t-spread* in $PG(d,q)$ is a set of disjoint t-dimensional subspaces. A partial t-spread in $PG(d,q)$ is *maximal* if it is not properly contained in any partial t-spread of $PG(d,q)$.

A *t-parallelism* of $PG(d,q)$ is a partition of the set of all t-dimensional subspaces by t-spreads. In the same way 1-parallelisms are simply called *parallelisms*.

The incidence of the points and t-dimensional subspaces of $PG(d,q)$ defines a 2-design (see for instance [77, 2.35-2.36]), i.e., the points of this design correspond to the points of the projective space, and its blocks to the t-dimensional subspaces. Therefore there is a one-to-one correspondence between t-parallelisms of $PG(d,q)$ and the resolutions of the related design of the points and t-dimensional subspaces.

An automorphism of $PG(d,q)$ is a bijective map on the point set that preserves collinearity, i.e., maps lines to lines, and thus t-dimensional subspaces to t-dimensional subspaces. Therefore all related designs have the full automorphism group of the projective space. Isomorphism and orthogonality of t-parallelisms are defined as for resolutions.

Baker [4], Beutelspacher [6], [7], Denniston [18], [19], [20], Johnson [27], [28], [29], [30], [31], [32], [33], [34], Prince [67], [68], Soicher [74], Storme [77] work hard on investigations of t-spreads and t-paralelisms.

Soicher [74] classified partial 1-spreads in finite projective spaces using the GRAPE package within the GAP system. His results include the construction up to equivalence of all partial 1-spreads in $PG(3,2)$, $PG(4,2)$ (verified without the help of computer by Shaw [25]) and $PG(3,3)$, all maximal partial 1-spreads in $PG(3,4)$ and the maximal partial 1-spreads in $PG(3,7)$ of size 45 and invariant under a group of order 5. Then Blokhuis, Brouwer and Wilbrink classified all maximal partial 1-spreads of size 45 in $PG(3,7)$ [8]. For odd q, Jha and Johnson [26] present a classification of all spreads in $PG(3,q)$ invariant under a certain group of automorphisms of order $q(q+1)$. Mateva and Topalova [59] classified up to equivalence all 1-spreads of $PG(5,2)$.

A construction of 1-parallelisms in $PG(d,2)$ is presented by Zaicev, Zinoviev and Semakov [97] and independently by Baker [4], and in $PG(2^n-1,q)$ by Beutelspacher [6]. Several constructions are known in $PG(3,q)$ due to Denniston [18], Johnson [27], [28], Johnson and Pomareda [34], Penttila and Williams [66].

Prince classifies 1-parallelisms of $PG(3,3)$ with automorphisms of order 5 [67] and 1-parallelisms of $PG(3,5)$ with automorphisms of order 31 (the cyclic parallelisms of $PG(3,5)$) [68]. Stinson and Vanstone classify 1-parallelisms of $PG(5,2)$ with a full automorphism group of order 155 [76], and Sarmiento with a point-transitive cyclic group of order 63 [70].

Transitive t-parallelisms have an automorphism group, which is transitive on their t-spreads [32]. Examples of transitive 1-parallelisms of $PG(3,q)$ are presented in [18], [66] and [68], and of $PG(5,2)$ in [76]. Transitivity and double transitivity are considered by Johnson [29] and by Johnson and Montinaro [33], who show that only two doubly-transitive parallelisms exist. They also point out in [33] that no examples of transitive t-parallelisms in $PG(d,q)$ are known for $t>1$, and the determining of parameters for which they exist is an open problem.

Sarmiento classifies 2-parallelisms of $PG(5,2)$ with an automorphism group, which is transitive on the points [71], i.e., a cyclic group of order 63. None of them has a group transitive on the 2-spreads.

Orthogonal t-parallelisms are of special interest [23], [76]. Stinson and Vanstone [76] apply an orthogonality test on the classified parallelisms and find a set of six mutually orthogonal 1-parallelisms of $PG(5,2)$. The point transitive 2-parallelisms constructed by Sarmiento [71] have a common 2-spread, and thus there are no orthogonal ones among them.

In the last chapter of the thesis we consider the 2-$(63,3,1)$ and 2-$(63,7,15)$ designs related to $PG(5,2)$. With some modification of our software we succeed in classifying resolutions with automorphisms of order 31 of these designs. They correspond to t-parallelisms of $PG(5,2)$ with automorphisms of order 31. The classified 1-parallelisms of $PG(5,2)$ are cyclic and together with those constructed by Stinson and Vanstone [76] present all cyclic 1-parallelisms of $PG(5,2)$. Part of the 2-parallelisms constructed in the thesis are examples of transitive ones.

There are 63 points, 651 lines and 1395 planes in $PG(5,2)$. A t-dimensional subspace has 2^{t+1}-1 points. The lines form a 2-$(63,3,1)$ design, and the 2-, 3-, and 4-dimensional subspaces a 2-$(63, 7, 15)$, 2-$(63, 15, 35)$, and 2-$(63, 31, 15)$ design respectively. Since the lines are 1-dimensional subspaces of $PG(5, 2)$, 1-parallelisms of $PG(5, 2)$ correspond to the resolutions of the 2-$(63, 3, 1)$ design and 1-spreads to its parallel classes. The resolutions and parallel classes of the 2-$(63, 7, 15)$ design correspond to 2-parallelisms and 2-spreads of $PG(5, 2)$.

The full automorphism group of $PG(5, 2)$ is isomorphic to the general linear group $GL(6, 2)$, $|GL(6, 2)| = 20158709760$.

We make all the computations on the related to $PG(5, 2)$ designs. Instead of a t-spread, we construct a parallel class, and instead of a parallelism, a resolution of the corresponding design. Also we find a generating set of the automorphism group of $PG(5, 2)$, as well as a subgroup of order 31 and its normalizer as automorphism groups of the related 2-$(63, 31, 15)$ point-hyperplane design.

We construct the 2-spreads of $PG(5, 2)$. To do this we use the specific properties of $GL(6, 2)$ and the subspaces of $PG(5, 2)$ to reduce the search space and to filter away equivalent solutions. The 2-spread elements are planes of $PG(5, 2)$. We choose the nine planes of a 2-spread among the 1395 blocks of the 2-$(63, 7, 15)$ design. For each pair of disjoint planes P_1 and P_2 and each 2-spread S of $PG(5, 2)$ there is an automorphism of $PG(5, 2)$ which transforms S in a 2-spread containing P_1 and P_2 as elements. This is why we only construct 2-spreads containing two fixed elements and therefore only part of the automorphisms of $PG(5, 2)$ can map the obtained 2-spreads to one another. We establish that the constructed 192 2-spreads are isomorphic. This coincides with [73, Theorem 4.1]. We find that the order of the automorphism group preserving the 2-spread is 10584. We later use our results about the 2-spread to obtain some of the 2-parallelisms of PG(5,2) in more than one way.

To construct t-parallelisms we use the most popular way of constructing $PG(5, 2)$ which follows from the vector space $V_6(2)$. The points of $PG(5, 2)$ correspond to the nonzero vectors of $V_6(2)$. The number of the point is the decimal value of the binary number corresponding to the vector. The points of the vectors of the $t + 1$-dimensional subspaces of $V_6(2)$ are in t-dimensional subspaces of $PG(5, 2)$. We then construct the related designs.

Since 31 divides the order of G, but 31^2 does not, by Sylow Theorems all subgroups of order 31 are conjugate, and we can choose an arbitrary one of them. Denote it by G_{31}. It is cyclic and fixes one point, while the other 62 points are in two orbits of length 31.

Let $\varphi \in G$ and R_1 be a resolution with automorphism group G_{R_1}. When we apply φ on R_1 we obtain another resolution $R_2 = \varphi R_1$ with automorphism group G_{R_2}. Let $\alpha \in G_{R_1}$ and $\beta \in G_{R_2}$. Then:

$$\begin{aligned} \beta\varphi R_1 &= \varphi\alpha R_1 \\ \beta &= \varphi\alpha\varphi^{-1} \\ G_{R_2} &= \varphi G_{R_1}\varphi^{-1}. \end{aligned}$$

We construct only the resolutions with the chosen automorphism group G_{31}. If R_1 and R_2 are two of the resolutions constructed by us it follows that $G_{R_2} = G_{R_1} = G_{31}$.

If an automorphism $\varphi \in G$ maps some of the constructed t-parallelisms to one another $G_{31} = \varphi G_{31}\varphi^{-1}$. Therefore to filter away isomorphic resolutions in our constructions we are interested in the normalizer $N(G_{31}) = \{g \in G \mid gG_{31}g^{-1} = G_{31}\}$ of G_{31} in G as $\varphi \in N(G_{31})$. The normalizer is a subgroup of order 155, let's denote it by G_{155}. Its generators are a nontrivial automorphism α from G_{31} and an automorphism φ of order 5.

Let $G_5 \cong G_{155}/G_{31}$ be the group generated by φ. Since we construct t-parallelisms of $PG(5,2)$ invariant under the group G_{31} we actually use the automorphism group G_5 to find all solutions orbits. We apply an automorphism from G_5 to each of the obtained solutions for t-parallelism and as result we can have a solution which is:

- lexicographically smaller than the current one, so it is skipped as already considered;
- lexicographically greater than the current one, so we apply the next automorphism of G_5. If all automorphisms lead to lexicographically greater solutions, the current one is nonisomorphic to the previously constructed ones;
- the same resolution, so its full automorphism group is of order 155, i.e., the order of $N(G_{31})$. The solution is nonisomorphic to the previously constructed.

A parallelism of $PG(5,2)$ consists of 31 spreads with 21 elements each. A parallelism is *cyclic* if there is an automorphism which permutes its spreads in one cycle. We construct the parallelisms with G_{31} and there are 31 spreads in a parallelism. They are cyclic parallelisms. Hence only one spread is enough to determine the whole parallelism. If we know it, we can use its orbit under G_{31} to obtain the other 30 spreads of the parallelism.

We construct the resolutions of the related 2-$(63, 3, 1)$ point-line design to obtain the parallelisms of $PG(5, 2)$. The 651 lines of $PG(5, 2)$ are the blocks of the corresponding 2-$(63, 3, 1)$ design. We order them lexicographically with respect to the numbers of the points they contain. G_{31} partitions all blocks into 21 orbits of length 31. We sort each block orbit with respect to the contained block numbers.

To obtain a parallel class 21 disjoint blocks are needed. A constructed class should have blocks from 21 different orbits under G_{31}. Without loss of generality the parallel class can contain the first line which is in the first orbit. From each of the remaining orbits we can delete the blocks incident with any point of the first block. In this way we use only part of the orbits (henceforth we call them shortened orbits).

We perform a backtrack search on the shortened orbits to choose the next 20 elements of a parallel class. If there are already n blocks in a parallel class, we choose the $n + 1$st one among the blocks with points which are in none of the n blocks. In this way we construct 5451326 resolutions of the 2-$(63, 3, 1)$ point-line design with G_{31}.

To filter away isomorphic resolutions we use G_5. It turns out that all 5451326 resolutions are in 1090208 orbits of length 5 and 286 orbits of length 1. The latter correspond to parallelisms with full automorphism group of order 155 (this result coincides with [76]). All nonisomorphic cyclic parallelisms of $PG(5, 2)$ invariant under the chosen automorphism group of order 31 number 1090494.

5451326

$\swarrow \quad G_5 \quad \searrow$

1090208 orbits of length 5	286 orbits of length 1

$\searrow \quad \swarrow$

1090494

nonisomorphic cyclic parallelisms of $PG(5, 2)$

Fig. 5. The orbits of cyclic parallelisms of $PG(5, 2)$ under G_5

A 2-parallelism of $PG(5, 2)$ consists of 155 2-spreads with 9 elements each. We construct the resolutions of the 2-$(63, 7, 15)$ design of the points and planes of $PG(5, 2)$ to obtain 2-parallelisms. G_{31} splits the resolution parallel classes in 5 orbits with a length of 31. It is enough to know a parallel class from each of these 5 orbits to form a resolution. Let's call the first parallel class from each orbit *orbit leader*.

The 1395 design blocks are considered in lexicographic order defined on the numbers of the points they contain. We apply the CC construction to obtain all possible parallel classes. Our software constructs only parallel classes beginning with one of the first 155 blocks (because only these blocks contain the first point). As a result we obtain 12288 parallel classes for a block $i, i = 1, \ldots, r$.

Next we apply G_{31} on them. At most 5678 parallel classes for each initial block remain. Each of these parallel classes has an orbit length of 31 under G_{31} and therefore can be an orbit leader. We use exhaustive backtrack search to choose 5 of them in order to obtain a design resolution. Suppose we have added $n < 5$ orbit leaders. We choose the next one such that it:

- contains the block with smallest number which is not yet in the resolution;
- has nine disjoint design blocks from nine different orbits (under G_{31}), which are not used yet in the resolution.

This way we construct 61192 resolutions with automorphisms of order 31. To filter away isomorphic resolutions we use the same manner as for resolutions of the 2-$(63, 3, 1)$ design. The obtained solutions are divided in two parts: 92 nonisomorphic resolutions with full automorphism group G_{155} and the remaining 61100 in 12220 orbits under G_5, i.e., 12220 non-isomorphic resolutions with full automorphism group G_{31}. In general there are 12312 non-isomorphic resolutions with automorphisms of order 31 (Fig. 6). These are the resolutions of the design on the points and planes of $PG(5, 2)$, therefore they are the 2-parallelisms of this projective space. G_{155} is transitive on the 2-spreads of the corresponding 2-parallelisms. Hence the obtained 92 2-parallelisms with full automorphism group of order 155 are transitive. They are the first example of transitive t-parallelisms with $t > 1$.

61192

$\swarrow \quad G_5 \quad \searrow$

12220 orbits of length 5   92 orbits of length 1

$\searrow \quad \swarrow$

12312

nonisomorphic 2-parallelisms of $PG(5, 2)$ with automorphisms of order 31

Fig. 6. 2-parallelisms of $PG(5, 2)$ with automorphisms of order 31

In the thesis the transitive resolutions with automorphisms of order 155 of the 2-(63,7,15) design on the points and planes of $PG(5,2)$ are investigated for mutual orthogonality. To obtain a set of m mutually orthogonal 2-parallelisms exhaustive backtrack search is used. We call a set maximal if no more 2-parallelisms can be added to it. The results are summarized in Table 3.

Table 3. Sets of m mutually orthogonal 2-parallelisms (m-MOPs) of $PG(5,2)$ with automorphism group of order 155

	maximal	all
2-MOPs	0	2308
3-MOPs	0	20602
4-MOPs	27	72504
5-MOPs	2234	106322
6-MOPs	9263	67549
7-MOPs	6930	19242
8-MOPs	1413	2563
9-MOPs	98	192
10-MOPs	10	10

Correctness of the computer results. The results obtained in the thesis have been partially verified in the following ways:

- Where possible the number of the combinatorial structures is found in two different ways.
- Some computer results are obtained by two different software implementations (of Topalova and Zhelezova).
- We have tested our software on parameters for which there are results announced by other authors and have obtained the same results.
- We also checked part of the results by other specialized programs, namely the computer system for algebraic computations GAP (`http://www.gap-system.org/`) and some programs by Mateva [58].

4. Conclusion. At the very beginning we outlined the ultimate goal—to obtain a complete classification of doubly resolvable designs and their sets of mutually orthogonal resolutions. In order to address this goal we defined three problems to solve in our paper:

- Develop a method for classification of doubly resolvable designs and implement it in our own programs.

- Classify doubly resolvable designs with small parameters and also their sets of mutually orthogonal resolutions.

- Find applications of the developed constructions to other combinatorial structures.

These problems were successfully addressed in the thesis and as a result we ended up with a method for classification of doubly resolvable designs and its software implementation. This is how we may outline the following major contributions of our solution:

- A classification of doubly resolvable 2-(v, k, λ) designs with small parameters and their sets of mutually orthogonal resolutions was made.

- The classification of cyclic parallelisms of $PG(5, 2)$ was completed.

- The 2-parallelisms of $PG(5, 2)$ with automorphisms of order 31 were classified and the first examples of transitive and orthogonal t-parallelisms for $t > 1$ were found among them.

The method for classification of resolutions which are orthogonal to at least one other resolution ($RORs$), the doubly resolvable designs (DRD) and their sets of mutually orthogonal resolutions (m-$MORs$) and the used orthogonal resolution existence (ORE) test constuctions were developed jointly with Topalova and published in [89], [91], [92], [93] and [95]. The derived restrictions on the doubly resolvable designs structure are published in [90], [98] and [99].

The classification of cyclic parallelisms of $PG(5, 2)$ is presented in [100]. The classification of 2-parallelisms of $PG(5, 2)$ with automorphisms of order 31 is joint work with Topalova and is published in [94].

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Stela Zhelezova
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: `stela@math.bas.bg`

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