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THE NONEXISTENCE OF $[132, 6, 86]_3$ CODES AND $[135, 6, 88]_3$ CODES

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Abstract. We prove the nonexistence of $[g_3(6,d), 6, d]_3$ codes for $d = 86, 87, 88$, where $g_3(k,d) = \sum_{i=0}^{k-1} \lceil d/3^i \rceil$. This determines $n_3(6,d)$ for $d = 86, 87, 88$, where $n_q(k,d)$ is the minimum length n for which an $[n,k,d]_q$ code exists.

1. Introduction. An $[n,k,d]_q$ code $\mathcal{C}$ is a linear code of length n, dimension k and minimum weight d over $\mathbb{F}_q$, the field of q elements. The *weight* of a vector $\boldsymbol{x} \in \mathbb{F}_q^n$, denoted by $wt(\boldsymbol{x})$, is the number of nonzero coordinate positions in $\boldsymbol{x}$. We only consider *non-degenerate* codes having no coordinate which is identically zero.

A fundamental problem in coding theory is to find $n_q(k,d)$, the minimum length n for which an $[n,k,d]_q$ code exists. See [8] for the updated tables of $n_q(k,d)$ for some small q and k. For ternary linear codes, $n_3(k,d)$ is known for $k \le 5$ for all d ([5]), but the value of $n_3(6,d)$ is still unknown for many integer d although the Griesmer bound is attained for all $d \ge 352$. It is known that

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$n_3(6,d) = g_3(6,d)$ or $g_3(6,d)+1$ for $d = 86, 87, 88$, where $g_3(k,d) = \sum_{i=0}^{k-1} \lceil d/3^i \rceil$ is the Griesmer bound, see [9]. An $[n,k,d]_q$ code attaining the Griesmer bound is called a *Griesmer code*. Our purpose is to prove the following theorems.

Theorem 1.1. *There exist no* $[132, 6, 86]_3$ *codes.*

Theorem 1.2. *There exist no* $[135, 6, 88]_3$ *codes.*

Corollary 1.3. $n_3(6,d) = g_3(6,d) + 1$ *for* $d = 86, 87, 88$.

The code obtained by deleting the same coordinate from each codeword of $\mathcal{C}$ is called a *punctured code* of $\mathcal{C}$. If there exist an $[n+1, k, d+1]_q$ code which gives $\mathcal{C}$ as a punctured code, $\mathcal{C}$ is called *extendable*. To prove Theorem 1.1, we show the that a putative $[132, 6, 86]_3$ code is extendable.

2. Preliminary results.

We denote by $\mathrm{PG}(r,q)$ the projective geometry of dimension r over $\mathbb{F}_q$. 0-flats, 1-flats, 2-flats, 3-flats, $(r-2)$-flats and $(r-1)$-flats are called *points, lines, planes, solids, secundums* and *hyperplanes* respectively. We denote by $\mathcal{F}_j$ the set of j-flats of $\mathrm{PG}(r,q)$ and by θ_j the number of points in a j-flat, i.e. $\theta_j = (q^{j+1}-1)/(q-1)$. We set $\theta_j = 0$ for $j < 0$.

Let $\mathcal{C}$ be a non-degenerate $[n,k,d]_q$ code. The columns of a generator matrix of $\mathcal{C}$ can be considered as a multiset of n points in $\Sigma = \mathrm{PG}(k-1,q)$ denoted also by $\mathcal{C}$. We see linear codes from this geometrical point of view. An *i-point* is a point of Σ which has multiplicity i in $\mathcal{C}$. Denote by γ_0 the maximum multiplicity of a point from Σ in $\mathcal{C}$ and let C_i be the set of i-points in Σ, $0 \le i \le \gamma_0$. For any subset S of Σ we define *the multiplicity of S with respect to $\mathcal{C}$*, denoted by $m_{\mathcal{C}}(S)$, as

$$m_{\mathcal{C}}(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|,$$

where $|T|$ denotes the number of elements in a set T. When the code is projective, i.e. when $\gamma_0 = 1$, the multiset $\mathcal{C}$ forms an n-set in Σ and the above $m_{\mathcal{C}}(S)$ is equal to $|\mathcal{C} \cap S|$. A line l with $t = m_{\mathcal{C}}(l)$ is called a *t-line*. A *t-plane*, a *t-solid* and so on are defined similarly. Then we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ such that $n = m_{\mathcal{C}}(\Sigma)$ and $n - d = \max\{m_{\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\}$. Conversely such a partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ as above gives an $[n,k,d]_q$ code in a natural manner. For an m-flat Π in Σ we define

$$\gamma_j(\Pi) = \max\{m_{\mathcal{C}}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\},\ 0 \le j \le m.$$

We write simply γ_j instead of $\gamma_j(\Sigma)$. It holds that $\gamma_{k-2} = n-d$, $\gamma_{k-1} = n$. When $\mathcal{C}$ is Griesmer, γ_j's are uniquely determined [6] as follows.

$$\gamma_j = \sum_{u=0}^{j} \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \text{ for } 0 \le j \le k-1. \tag{2.1}$$

Hence, every Griesmer $[n,k,d]_q$ code is projective if $d \le q^{k-1}$. In this paper, we only consider projective codes. Denote by a_i the number of hyperplanes Π in Σ with $m_{\mathcal{C}}(\Pi) = i$. The list of a_i's is called the *spectrum* of $\mathcal{C}$. We usually use τ_j's for the spectrum of a hyperplane of Σ to distinguish from the spectrum of $\mathcal{C}$. A simple counting of argument yields the following.

Lemma 2.1. *A projective* $[n,k,d]_q$ *code satisfies*

$$(1) \sum_{i=0}^{n-d} a_i = \theta_{k-1}, \quad (2) \sum_{i=1}^{n-d} i a_i = n\theta_{k-2}, \quad (3) \sum_{i=2}^{n-d} i(i-1)a_i = n(n-1)\theta_{k-3}.$$

We get the following from the three equalities of Lemma 2.1:

$$\sum_{i=0}^{n-d-2} \binom{n-d-i}{2} a_i = \binom{n-d}{2}\theta_{k-1} - n(n-d-1)\theta_{k-2} + \binom{n}{2}\theta_{k-3}. \tag{2.2}$$

Lemma 2.2 ([10])**.** *Let* Π *be an* i*-hyperplane through a* t*-secundum* δ*. Then*

(1) $t \le \gamma_{k-2} - (n-i)/q = (i + q\gamma_{k-2} - n)/q$.

(2) $a_i = 0$ *if an* $[i, k-1, d_0]_q$ *code with* $d_0 \ge i - \lfloor (i + q\gamma_{k-2} - n)/q \rfloor$ *does not exist, where* $\lfloor x \rfloor$ *denotes the largest integer less than or equal to* x*.*

(3) $\gamma_{k-3}(\Pi) = \lfloor (i + q\gamma_{k-2} - n)/q \rfloor$ *if an* $[i, k-1, d_1]_q$ *code satisfying* $d_1 \ge i - \lfloor (i + q\gamma_{k-2} - n)/q \rfloor + 1$ *does not exist.*

(4) *Let* c_j *be the number of* j*-hyperplanes through* δ *other than* Π*. Then*

$$\sum_j (\gamma_{k-2} - j)c_j = i + q\gamma_{k-2} - n - qt. \tag{2.3}$$

(5) *For a* γ_{k-2}*-hyperplane* Π_0 *with spectrum* $(\tau_0, \ldots, \tau_{\gamma_{k-3}})$*,* $\tau_t > 0$ *holds if* $i + q\gamma_{k-2} - n - qt < q$*.*

Theorem 2.3 ([11])**.** *Let* $\mathcal{C}$ *be a Griesmer* $[n,k,d]_p$ *code,* p *a prime. If* p^e *divides* d*, then* p^e *is a divisor of all nonzero weights of* $\mathcal{C}$*.*

Let $\mathcal{C}$ be an $[n,k,d]_3$ code with $k \geq 3$, $\gcd(3,d) = 1$. The *diversity* (Φ_0, Φ_1) of $\mathcal{C}$ was defined in [12] as the pair of integers:

$$\Phi_0 = \frac{1}{2} \sum_{3|n-i} a_i, \quad \Phi_1 = \frac{1}{2} \sum_{i \not\equiv n, n-d \pmod 3} a_i,$$

where the notation $x|y$ means that x is a divisor of y. Let

$$\begin{aligned} F_0 &= \{\pi \in \mathcal{F}_{k-2} \mid m_{\mathcal{C}}(\pi) \equiv n \pmod 3\}, \\ F_1 &= \{\pi \in \mathcal{F}_{k-2} \mid m_{\mathcal{C}}(\pi) \not\equiv n, n-d \pmod 3\}, \\ F_2 &= \{\pi \in \mathcal{F}_{k-2} \mid m_{\mathcal{C}}(\pi) \equiv n-d \pmod 3\}. \end{aligned}$$

Then we have $\Phi_s = |F_s|$ for $s = 0, 1$.

The diversity can be applied to the dual space Σ^* of Σ. A t-flat Π of Σ^* with $|\Pi \cap F_0| = i$, $|\Pi \cap F_1| = j$ is called an $(i,j)_t$ *flat*. An $(i,j)_1$ flat is called an (i,j)-*line*. An (i,j)-*plane*, an (i,j)-*solid* and so on are defined similarly. We denote by $\mathcal{F}_j^*$ the set of j-flats of Σ^*. Let Λ_t be the set of all possible (i,j) for which an $(i,j)_t$ flat exists in Σ^*. Then we have

$$\begin{aligned} \Lambda_1 &= \{(1,0),(0,2),(2,1),(1,3),(4,0)\}, \\ \Lambda_2 &= \{(4,0),(1,6),(4,3),(4,6),(7,3),(4,9),(13,0)\}, \\ \Lambda_3 &= \{(13,0),(4,18),(13,9),(10,15),(16,12),(13,18),(22,9),(13,27),(40,0)\}, \\ \Lambda_4 &= \{(40,0),(13,54),(40,27),(31,45),(40,36),(40,45),(49,36),(40,54),(67,27), \\ &\quad (40,81),(121,0)\}, \\ \Lambda_5 &= \{(121,0),(40,162),(121,81),(94,135),(121,108),(112,126),(130,117), \\ &\quad (121,135),(148,108),(121,162),(202,81),(121,243),(364,0)\}, \end{aligned}$$

see [12]. Let $\Pi_t \in \mathcal{F}_t$. Let ${\varphi_s}^{(t)} = |\Pi_t \cap F_s|$, $s = 0,1$. $({\varphi_0}^{(t)}, {\varphi_1}^{(t)})$ is called the *diversity of* Π_t.

We use the following extension theorem to prove Theorem 1.1.

Theorem 2.4 ([3]). *Let $\mathcal{C}$ be an $[n,k,d]_3$ code with gcd$(d,3) = 1$ whose diversity satisfies $\Phi_1 = 0$. Then $\mathcal{C}$ is extendable.*

The following Lemma gives the set of all possible diversities of non-extendable $[n,k,d]_3$ codes for $k = 5,6$, which is needed later.

Lemma 2.5 ([7]). *Let $\mathcal{C}$ be an $[n,k,d]_3$ code with diversity (Φ_0, Φ_1), gcd$(3,d) = 1$. If $\mathcal{C}$ is not extendable, then*

(1) *when* $k = 5$, $(\Phi_0, \Phi_1) \in \mathcal{D}_5^+ = \{(40, 27), (31, 45), (40, 36), (40, 45), (49, 36)\}$,
(2) *when* $k = 6$, $(\Phi_0, \Phi_1) \in \mathcal{D}_6^+ = \{(121, 81), (94, 135), (121, 108), (112, 126), (130, 117), (121, 135), (148, 108)\}$.

The following Lemmas 2.6 and 2.7 can be derived from Theorems 3.12, 3.13, 3.16 in [12].

Lemma 2.6 ([12]). *Let* Π *be a* $(\varphi_0, \varphi_1)_4$ *flat with* $(\varphi_0, \varphi_1) \in \mathcal{D}_5^+$.
(1) *For any point* P *of* $F_0 \cap \Pi$, *the numbers of* (i, j)*-lines through* P *in* Π, *denoted by* $p_{i,j}$, *is as in* Table 2.1.

Table 2.1

φ_0	φ_1	$p_{1,0}$	$p_{2,1}$	$p_{4,0}$	$p_{1,3}$
40	27	18	0	13	9
		9	27	4	0
31	45	15	0	10	15
		6	27	1	6
40	36	6	27	4	3
40	45	3	27	4	6
49	36	12	0	16	12
		3	27	7	3

(2) *For any point* Q *of* $F_1 \cap \Pi$, *the numbers of* (i, j)*-lines through* Q *in* Π, *denoted by* $q_{i,j}$, *is as in* Table 2.2.

Table 2.2

φ_0	φ_1	$q_{1,3}$	$q_{0,2}$	$q_{2,1}$
40	27	4	18	18
31	45	13	18	9
40	36	10	15	15
40	45	16	12	12
49	36	13	9	18

Table 2.3

φ_0	φ_1	$r_{1,0}$	$r_{2,1}$	$r_{0,2}$
40	27	22	9	9
31	45	13	9	18
40	36	16	12	12
40	45	10	15	15
49	36	13	18	9

(3) *For any point* R *of* $F_2 \cap \Pi$, *the numbers of* (i, j)*-lines through* R *in* Π, *denoted by* $r_{i,j}$, *is as in* Table 2.3.

Lemma 2.7 ([12]). *Let* Π *be a* $(\varphi_0, \varphi_1)_5$ *flat with* $(\varphi_0, \varphi_1) \in \mathcal{D}_6^+$.
(1) *For any point* Q *of* $F_1 \cap \Pi$, *there are at most* 54 $(2, 1)$*-lines through* Q *in* Π.
(2) *For any point* R *of* $F_2 \cap \Pi$, *the number of* $(1, 0)$*-lines or* $(0, 2)$*-lines through* R *in* Π *is at most* 94.

3. Spectra of some $[n,k,d]_3$ codes. In this section, we give some results on ternary linear codes, which are needed in the next sections. Table 3.1 can be obtained from the known results [2].

Table 3.1. The spectra of some ternary linear codes.

parameters	possible spectra
$[7,4,3]_3$	$(a_1,a_2,a_3,a_4)=(14,9,9,8)$
	$(a_0,a_1,a_2,a_3,a_4)=(2,9,12,10,7)$
	$(a_0,a_1,a_2,a_3,a_4)=(4,4,15,11,6)$
	$(a_0,a_1,a_2,a_3,a_4)=(3,8,9,15,5)$
$[8,4,4]_3$	$(a_0,a_1,a_2,a_3,a_4)=(3,4,10,12,11)$
	$(a_0,a_2,a_3,a_4))=(4,16,8,12)$
	$(a_0,a_1,a_2,a_3,a_4)=(2,8,4,16,10)$
$[9,4,5]_3$	$(a_0,a_1,a_3,a_4)=(1,9,12,18)$
$[15,4,9]_3$	$(a_3,a_6)=(15,25)$
	$(a_0,a_3,a_6)=(1,13,26)$
$[10,5,5]_3$	$(a_1,a_2,a_4,a_5)=(10,45,30,36)$
$[16,5,9]_3$	$(a_1,a_4,a_7)=(6,57,58)$
$[19,5,11]_3$	$(a_1,a_2,a_4,a_5,a_7,a_8)=(1,9,9,27,30,45)$

Lemma 3.1 ([2])**.** *The spectrum of a projective* $[15,4,9]_3$ *code is* $(a_3,a_6)=(15,25)$.

The following information about the classification of some ternary codes was supplied by I. Bouyukliev via T. Maruta.

Lemma 3.2 (cf.[1])**.**

(1) *The spectrum of a* $[25,5,15]_3$ *code is either* $(a_1,a_4,a_7,a_{10})=(1,12,43,65)$ *or* $(a_4,a_7,a_{10})=(15,40,66)$.

(2) *The spectrum of a projective* $[28,5,17]_3$ *code is* $(a_1,a_5,a_8,a_{10},a_{11})=(1,18,18,39,45)$.

(3) *The spectrum of a* $[37,5,23]_3$ *code is either* $(a_1,a_8,a_{10},a_{11},a_{13},a_{14})=(1,18,9,9,30,54)$ *or* $(a_2,a_7,a_8,a_{10},a_{11},a_{13},a_{14})=(1,4,14,5,13,31,53)$.

(4) *The spectrum of a* $[47,5,30]_3$ *code is either* $(a_5,a_8,a_{11},a_{14},a_{17})=(1,4,10,23,83)$ *or* $(a_2,a_{11},a_{14},a_{17})=(1,18,18,84)$.

(5) *Every* $[29,5,18]_3$ *code is not projective.*

Lemma 3.3. *Every* $[46,5,29]_3$ *code is extendable.*

Proof. Let $\mathcal{C}$ be a $[46,5,29]_3$ code and let Δ be a γ_3-solid, which gives a $[17,4,10]_3$ by Lemma 2.2. So we have $a_3 = a_6 = a_{12} = 0$ by Lemma 2.2 and the known $n_3(4,d)$ table. Now, $F_1 = \{$0-solids, 9-solids, 15-solids$\}$. From (2.2), we obtain

$$\begin{aligned} &136a_0 + 120a_1 + 105a_2 + 78a_4 + 66a_5 + 45a_7 + 36a_8 \\ &\quad +28a_9 + 21a_{10} + 15a_{11} + 6a_{13} + 3a_{14} + a_{15} = 471 \end{aligned} \tag{3.1}$$

since $\mathcal{C}$ is projective. And a 15-solid in Σ =PG(4, 3) gives a $[15,4,9]_3$ code by Lemma 2.2, which is also projective. Hence it has only 3-planes or 6-planes by Lemma 3.1.

Suppose $a_0 > 0$ and let Δ_1 be a 0-solid in Σ. For $i = 0$, the maximum possible contribution of c_j's in (2.3) to the LHS of (3.1) is $(c_{13}, c_{16}, c_{17}) = (1,1,1)$ for $t = 0$. Estimating the LHS of (3.1) we get $471 \le 6 \cdot 40 + 136 = 376$, a contradiction. Hence $a_0 = 0$.

Now, $\mathcal{C}$ is not extendable by (4) of Lemma 3.2 if $a_9 + a_{15} > 0$. Then, the diversity (Φ_0, Φ_1) of $\mathcal{C}$ satisfies $(\Phi_0, \Phi_1) \in \mathcal{D}_5^+$ by Lemma 2.5.

Suppose $a_9 > 0$. Let Δ_2 be a 9-solid in Σ and let Δ_2^* be the corresponding point of F_1 in Σ^*. Then Δ_2 gives a $[9,4,5]_3$ code by Lemma 2.2. Hence the spectrum of Δ_2 is $(\tau_0, \tau_1, \tau_3, \tau_4) = (1,9,12,18)$. For $i = 9, t = 4$, the equation (2.3) has the unique solution $(c_{16}, c_{17}) = (2,1)$ corresponding to a $(2,1)$-line through Δ_2^*. And for $i = 9$, a t-solid with the solution of (2.3) corresponding to a $(1,3)$-line exists only when $t = 3$, because a 15-solid in Σ has only 3-planes or 6-planes. Hence, there are at least $\tau_4 = 18$ $(2,1)$-lines through Δ_2^* and there are at most $\tau_3 = 12$ $(1,3)$-lines through Δ_2^*. Therefore $(\Phi_0, \Phi_1) = (40,27)$, $\gamma_{1,3} = 4$, $\gamma_{0,2} = 18$, $\gamma_{2,1} = 18$ by Table 2.2, where $\gamma_{i,j}$ denotes the number of (i,j)-lines through Δ_2^* in Σ^*. And then one 0-plane and nine 1-planes, eight 3-planes in Δ_1 correspond to $(0,2)$-lines through Δ_2^* in Σ^*. For $i = 9, t = 0,1,3$ in Lemma 2.2, the equation (2.3) has the solution corresponding to a $(0,2)$-line as Table 3.2. Hence, estimating the LHS of (3.1) we get $471 \le 43 \cdot 1 + 31 \cdot 9 + 4 \cdot 8 + 2 \cdot 4 + 28 = 390$, a contradiction. Thus $a_9 = 0$.

Suppose $a_{15} > 0$ and $a_7 > 0$. Let π_1 be a 7-solid in Σ and let P be the corresponding point of F_0 in Σ^*. Then π_1 gives a $[7,4,3]_3$ code by Lemma 2.2. Hence the spectrum of π_1 satisfies $\tau_3 \le 15$. For $i = 7$, the equation (2.3) has no solution corresponding to a $(1,3)$-line through P in Σ^* and a t-solid with the solution of (2.3) corresponding to a $(2,1)$-line exists only when $t = 3$, since a 15-solid in Σ has only 3-planes or 6-planes. Hence, $\gamma_{1,3} = 0$, $\gamma_{2,1} \le 15$. But there exists no diversity satisfying this condition in Table 2.1, a contradiction. Thus $a_{15} > 0$ implies $a_7 = 0$.

Next, suppose $a_{15} > 0$ and $a_8 > 0$. Let π_2 be a 8-solid in Σ and let R be the corresponding point of F_2 in Σ^*. Then π_2 gives a $[8, 4, 4]_3$ code by Lemma 2.2, and the spectrum of π_2 satisfies $\tau_3 \leq 16$. For $i = 8$, a t-solid with the solution of (2.3) corresponding to a $(0, 2)$-line or a $(2, 1)$-line through R exists only when $t = 3$, since a 15-solid in Σ has only 3-planes or 6-planes. Hence, $\gamma_{0,2} + \gamma_{2,1} \leq 16$, contradicting Table 2.3. Hence, $a_{15} > 0$ implies that $a_7 = a_8 = 0$.

Suppose $a_{15} > 0$. Since $\mathcal{C}$ is projective, the spectrum of a 15-solid is $(\tau_3, \tau_6) = (15, 25)$ by Lemma 3.1. Then, for $i = 15$, the maximum possible contributions of c_j's in (2.3) to the LHS of (3.1) are $(c_{10}, c_{13}, c_{17}) = (1, 1, 1)$ for $t = 3$ and $(c_{15}, c_{17}) = (1, 2)$ for $t = 6$, since $a_7 = a_8 = 0$. Estimating the LHS of (3.1) we get $471 \leq 27 \cdot 15 + 1 \cdot 25 + 1 = 431$, a contradiction. Hence $a_{15} = 0$.

Now, our assertion follows from Theorem 2.4. □

Table 3.2. Solutions of (2.3) for $i = 9$ corresponding to a $(0, 2)$-line

t	c_1	c_2	c_4	c_5	c_7	c_8	c_9	c_{10}	c_{11}	c_{13}	c_{14}	c_{15}	c_{16}	c_{17}
0							1		1			–		1
							1				2	–		
1							1				1	–		1
3											1	1		1

Corollary 3.4. *The spectrum of a* $[46, 5, 29]_3$ *code satisfies*

$$a_i = 0 \text{ for all } i \notin \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17\}.$$

4. Proof of Theorems 1.1 and 1.2.

Lemma 4.1. *There exists no* $[133, 6, 87]_3$ *code.*

Proof. Let $\mathcal{C}$ be a putative $[133, 6, 87]_3$ code and let Π be a γ_4-hyperplane in $\Sigma = \mathrm{PG}(5, 3)$. Then Π satisfies $\tau_i = 0$ for all $i \notin \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17\}$ by Corollary 3.4, so $a_i = 0$ for all $i \notin \{1, 10, 16, 19, 25, 28, 34, 37, 43, 46\}$ by Theorem 2.3, Lemma 2.2 and the known $n_3(5, d)$-table. From (2.1), we obtain

$$35a_1 + 22a_{10} + 15a_{16} + 12a_{19} + 7a_{25} + 5a_{28} + 2a_{34} + a_{37} = 112 \tag{4.1}$$

since $\mathcal{C}$ is projective.

Suppose $a_1 > 0$ and let π_1 be a 1-hyperplane. The spectrum of π_1 is $(\tau_0, \tau_1) = (81, 40)$. Then the solutions of (2.3) for $i = 1$ are $(c_{43}, c_{46}) = (2, 1)$ for $t = 0$ and $(c_{43}, c_{46}) = (1, 2)$ for $t = 1$. Hence $a_1 = 1$ and $a_i = 0$ for $10 \le i \le 37$, contradicting (4.1). Thus $a_1 = 0$.

Suppose $a_{10} > 0$ and let π_2 be a 10-hyperplane. Then π_2 gives a $[10, 5, 5]_3$ code by Lemma 2.2. The spectrum of π_2 is $(\tau_1, \tau_2, \tau_4, \tau_5) = (10, 45, 30, 36)$ by Table 3.1. For $i = 10$, the maximum possible contributions of c_j's in (2.3) to the LHS of (4.1) are $(c_{34}, c_{46}) = (1, 2)$ for $t = 1$ and $(c_{37}, c_{46}) = (1, 2)$ for $t = 2$ and $(c_{43}, c_{46}) = (1, 2)$ for $t = 4$ and $c_{46} = 3$ for $t = 5$. Estimating the LHS of (4.1) we get $112 \le 2 \cdot 10 + 1 \cdot 45 + 22 = 87$, a contradiction. Hence $a_{10} = 0$.

Similarly, for $i = 16, 19$, considering the maximum possible contributions of c_j's in (2.3) to the LHS of (4.1) gives a contradiction. Hence $a_{16} = a_{19} = 0$.

Suppose $a_{25} > 0$ and let π_3 be a 25-hyperplane. Then π_3 gives a $[25, 5, 15]_3$ code by Lemma 2.2. Hence there are two possible spectra for π_5 by Lemma 3.2. We first assume that the spectrum of π_3 is $(\tau_1, \tau_4, \tau_7, \tau_{10}) = (1, 12, 43, 65)$. For $i = 25$, the maximum possible contributions of c_j's in (2.3) to the LHS of (4.1) are $(c_{25}, c_{43}) = (1, 2)$ for $t = 1$ and $(c_{34}, c_{43}) = (1, 2)$ for $t = 4$ and $(c_{37}, c_{46}) = (1, 2)$ for $t = 7$ and $c_{46} = 3$ for $t = 10$, since $c_{28} = 0$ when $t = 4$ by Lemma 3.2. Estimating the LHS of (4.1) we get $112 \le 7 \cdot 1 + 2 \cdot 12 + 1 \cdot 43 + 7 = 81$, a contradiction. We get a contradiction similarly for the other spectrum of π_5. Hence $a_{25} = 0$.

Suppose $a_{28} > 0$ and let π_4 be a 28-hyperplane. Then π_4 gives a $[28, 5, 17]_3$ code by Lemma 2.2. The spectrum of π_4 is $(\tau_1, \tau_5, \tau_8, \tau_{10}, \tau_{11}) = (1, 18, 18, 39, 45)$ by Lemma 3.2. For $i = 28$, the equation (2.3) has the solutions as in Table 4.1.

Table 4.1

t	c_{28}	c_{34}	c_{37}	c_{43}	c_{46}
1	1	1			1
	1		1	1	
		1	2		
5	1		–		2
		1	–	2	
8			1		2
				3	
10				1	2
11					3

Since there are one 1-solid and 18 5-solids in π_4, we get $a_{28} + a_{34} \le \tau_1 \cdot 2 + \tau_5 \cdot 1 = 20$

by Table 4.1. Similarly, we also get $a_{28}+a_{34}+a_{37} \le \tau_1 \cdot 3+\tau_5 \cdot 1+\tau_8 \cdot 1 = 39$ by Table 4.1. Hence, we get $73-4a_{28} \le a_{34} \le 20-a_{28}$ from these two inequalities and (4.1). Hence, if $a_{28} > 0$, then it holds that $a_{28} \ge 18$ and there exists a 46-hyperplane which has a 5-solid. (Otherwise, estimating the maximum possible LHS of (4.1) we get $112 \le 7 \cdot 1 + 2 \cdot 18 + 1 \cdot 18 + 5 = 66$, a contradiction.) Now, let Π' be a 46-hyperplane containing a 5-solid. Then Π' gives a $[46,5,29]_3$ code by Lemma 2.2. For $i = 46$, the solutions of the equation (2.3) satisfy $c_{28} \le 2$ when $t = 5$ and $c_{28} \le 1$ when $7 \le t \le 11$. Since the spectrum of Π' satisfies $\tau_5 = 1, \tau_7+\tau_8 = 4, \tau_{10}+\tau_{11} = 10$ by Lemma 3.3 and Lemma 3.2 (4), we get $a_{28} \le \tau_5 \cdot 2 + (\tau_7+\tau_8)\cdot 1 + (\tau_{10}+\tau_{11}) \cdot 1 = 16$, a contradiction. Hence $a_{28} = 0$. Now, we get $2a_{34}+a_{37} = 112$ from (4.1), and $4a_{34}+3a_{37}+a_{43} = 217$ from (1) and (2) of Lemma 2.1, whence $a_{37}+a_{43} = -7$, a contradiction. This completes the proof. □

P r o o f o f T h e o r e m 1.1. Let $\mathcal{C}$ be a putative $[132,6,86]_3$ code and let Π be a γ_4-hyperplane in Σ =PG(5,3). Then Π satisfies $\tau_i = 0$ for all $i \notin \{1,2,4,5,7,8,10,11,13,14,16,17\}$ by Corollary 3.4, so $a_i = 0$ for all $i \notin \{0-2,6,9-11,15,16,18-20,24,25,27-29,33,34,36-38,42,43,45,46\}$ by Lemma 2.2 and the known $n_3(5,d)$-table. Suppose $a_{29} > 0$ and let π_1 be a 29-hyperplane in Σ. Then π_1 gives a $[29,5,18]_3$ code by Lemma 2.2. By Lemma 3.2 (5), $\gamma_0(\pi_1) \ne 1$, which contradicts the fact that $\mathcal{C}$ is projective. Hence $a_{29} = 0$.
Since $\mathcal{C}$ is not extendable by Lemma 4.1, the diversity (Φ_0,Φ_1) of $\mathcal{C}$ satisfies $(\Phi_0,\Phi_1) \in \mathcal{D}_6^+$ by Lemma 2.5. And it holds that $F_1 = \{i\text{-hyperplanes} \mid i \in \{2,11,20,38\}\}$. Let π be an i-hyperplane in F_1 and let π^* be the point of F_1 in Σ^* corresponding to π. Then there are at most 54 (2,1)-lines through π^* in Σ^* by (1) of Lemma 2.7. If $i = 2$, π has 54 0-solids, 54 1-solids and 13 2-solids. Setting $i = 2$ in Lemma 2.2, the equation (2.3) has the unique solution $(c_{42},c_{43},c_{45}) = (1,1,1)$ corresponding to a (2,1)-line through π^* for $t = 0$, and $(c_{45},c_{46}) = (2,1)$ corresponding to a (2,1)-line through π^* for $t = 2$. Hence, there are at least 67 (2,1)-lines through π^*, a contradiction. Similarly, we can get a contradiction for $i = 11,20,38$. Thus $a_2 = a_{11} = a_{20} = a_{38} = 0$.

Hence we get $\Phi_1 = |F_1| = 0$, which implies that $\mathcal{C}$ is extendable by Theorem 2.4. But there exists no $[133,6,87]_3$ code by Lemma 4.1, a contradiction. This completes the proof. □

Lemma 4.2 ([4])**.** *There exists no* $[136,6,89]_3$ *code.*

P r o o f o f T h e o r e m 1.2. Let $\mathcal{C}$ be a putative $[135,6,88]_3$ code and let Π be a γ_4-hyperplane in Σ =PG(5,3). Then Π gives a $[47,5,30]_3$ code by Lemma 2.2. By Lemma 3.2, Π satisfies $\tau_i = 0$ for all $i \notin \{2,5,8,11,14,17\}$, so

$a_i = 0$ for all $i \notin \{2, 9\text{-}11, 18\text{-}20, 27\text{-}29, 36\text{-}38, 45\text{-}47\}$ by Lemma 2.2 and the known $n_3(5, d)$-table. Since $\mathcal{C}$ is not extendable by Lemma 4.2, the diversity (Φ_0, Φ_1) of $\mathcal{C}$ satisfies $(\Phi_0, \Phi_1) \in \mathcal{D}_6^+$ by Lemma 2.5. Now, let Σ^* be the dual space of Σ. Let Π^* be the point of F_2 corresponding Π in Σ^* and let $r_{i,j}$ be the number of (i, j)-lines through Π^*. Then, for $i = 47, t = 14, 17$, the equation (2.3) has no solution corresponding to a $(2, 1)$-line through Π^*. Thus, by Lemma 3.2, we get

$$r_{1,0} + r_{0,2} \geq \tau_{14} + \tau_{17} \geq 102,$$

which contradicts (2) of Lemma 2.7. This completes the proof. □

REFERENCES

[1] Bouyukliev I., J. Simonis. Some new results for optimal ternary linear codes. *IEEE Trans. Inform. Theory* **48** (2002), 981–985.

[2] Eupen M. van, P. Lisonek. Classification of some optimal ternary linear codes of small length. *Des. Codes Cryptogr.* **10** (1997), 63–84.

[3] Hill R. An extension theorem for linear codes. *Des. Codes Cryptogr.* **17** (1999), 151–157.

[4] Jones C. M. Optimal ternary linear codes. PhD thesis, University of Salford, 2000.

[5] Landjev I. N. The nonexistence of some optimal ternary linear codes of dimension five. *Des. Codes Cryptogr.* **15** (1998), 245–258.

[6] Maruta T. On the nonexistence of q-ary linear codes of dimension five. *Des. Codes Cryptogr.* **22** (2001), 165–177.

[7] Maruta T. Extendability of ternary linear codes. *Des. Codes Cryptogr.* **35** (2005), 175–190.

[8] Maruta T. Griesmer bound for linear codes over finite fields. `http://www.geocities.jp/mars39geo/griesmer.htm`.

[9] Maruta T., Y. Oya. On optimal ternary linear codes of dimension 6. *Adv. Math. Commun.*, (to appear).

[10] Takenaka M., K. Okamoto, T. Maruta. On optimal non-projective ternary linear codes. *Discrete Math.* **308** (2008), 842–854.

[11] Ward H. N. Divisibility of codes meeting the Griesmer bound. *J. Combin. Theory Ser. A* **83** (1998), 79–93.

[12] Yoshida Y. , T. Maruta. Ternary linear codes and quadrics. *Electronic J. Combin.* **16** (2009) #R9.

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