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ONE-PARAMETER BIFURCATION ANALYSIS OF DYNAMICAL SYSTEMS USING *MAPLE*[*]

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Abstract. This paper presents two algorithms for one-parameter local bifurcations of equilibrium points of dynamical systems. The algorithms are implemented in the computer algebra system *Maple 13*© and designed as a package. Some examples are reported to demonstrate the package's facilities.

1. Introduction. Nonlinear dynamical systems depending on parameters may have very complicated behavior. If the parameters are varied, the phase portrait may deform slightly without altering its qualitative (topological) features, or sometimes the dynamics may be modified significantly, producing a qualitative change in the phase portrait [1], [2], [4]. Bifurcation theory studies

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these qualitative changes in the phase portrait, e. g. the appearance or disappearance of equilibrium points, periodic orbits or more complicated features.

Consider the dynamical system

$$\dot{x} = f(x,p), \quad x \in R^n, \;\; p \in R^1, \tag{1}$$

where f is a smooth vector function. Suppose an asymptotically stable equilibrium (x^*,p) is perturbed by varying the system parameter p, and at a critical parameter value $p = p^*$ the equilibrium becomes nonhyperbolic, i. e., some eigenvalues of the linearization (Jacobian matrix) $D_x f(x,p)$ evaluated at (x^*,p^*) cross the imaginary axis. The question is what happens to the system as p is varied about p^*. This question can be answered using the *center manifold theory* and the method of *normal forms*, which will be shortly presented below.

Without loss of generality let us assume that $(x^*,p^*) = (0,0)$. Assume further that $D_x f(0,0)$ has n_0 eigenvalues with zero real part, and $n-n_0$ eigenvalues with nonzero real parts. Then (1) can be written in the following extended form

$$\begin{array}{rcll} \dot{u} &=& Au + F(u,v,p), & F(0,0,0)=0,\; DF(0,0,0)=0 \\ \dot{v} &=& Bv + G(u,v,p), & G(0,0,0)=0,\; DG(0,0,0)=0 \\ \dot{p} &=& 0, & \end{array} \tag{2}$$

where the parameter p is introduced as a new phase variable, A is an $n_0 \times n_0$ matrix having eigenvalues with zero real parts, B is an $(n-n_0)\times(n-n_0)$ matrix having eigenvalues with nonzero (negative and/or positive) real parts. The matrices A and B do not depend on the parameter p. An invariant manifold is called a center manifold for (2) if it can locally be represented as follows [4]

$$W^c(0) = \{(u,v,p) : \; v = V(u,p), \; |u| < \delta_1, \; |p| < \delta_2, \; V(0,0) = 0, \; DV(0,0) = 0\}$$

for δ_1 and δ_2 sufficiently small. The conditions $V(0,0) = 0$ and $DV(0,0) = 0$ imply that $W^c(0)$ is tangent at $(u,v,p) = (0,0,0)$ to the invariant subspace E^c spanned by the generalized eigenvectors, which correspond to the n_0 eigenvalues with zero real part. Then the dynamical system (2) is locally topologically equivalent with the system [1]

$$\dot{u} = Au + F(u,V(u,p),p), \quad (u,p) \in R^{n_0} \times R^1 \tag{3}$$

$$\dot{p} = 0 \tag{4}$$

$$\dot{v} = Bv, \quad v \in R^{n-n_0}. \tag{5}$$

The equations for u and v are uncoupled in the above system. The first two equations, (3) and (4), represent the restriction of (2), or equivalently of (1), on its center manifold. Since (5) is linear and has exponentially decaying/growing solutions, the analysis of bifurcations of the equilibrium points of (1) reduces to that of the restricted equations (3) and (4).

The computation of the center manifold (i. e., of the function $v = V(u, p)$) is a difficult problem. Fortunately there is a method [1], [4], based on power series expansions, for computing approximations of the center manifold to any desired degree of accuracy. In the next sections we shall show that quadratic approximation of V will suffice to determine the local steady states bifurcations and the Andronov-Hopf bifurcation.

Having the reduced dynamical system on its center manifold, the next goal is to simplify the nonlinear part F in (3), yielding the so-called topological normal form of the bifurcation.

The analytical parameter and coordinate transformations required to put the system into its topological normal form lead to lengthy intermediate calculations like symbolic Jacobian computations, Taylor series coefficients, eigenvalues and eigenvectors computations. The natural environment for this kind of work are the computer algebra systems (CAS) like *Maple* and *Mathematica*. Their impact on dynamical systems studies is due to the fact that many calculations are too tedious for manual work, but do not challenge the computer resources [3].

In this paper we present two algorithms for symbolical study of one-parameter local bifurcations of equilibrium points and discuss their implementation in CAS *Maple*.

2. Local bifurcations of equilibrium points with single zero eigenvalue. Consider the dynamical system (1) and assume that (x^*, p^*) is a nonhyperbolic equilibrium point with a single zero eigenvalue of the linearization $D_x f(x^*, p^*)$, with the remaining eigenvalues having nonzero real parts. Below we present the main steps of the algorithm for normal form computation of this type of bifurcations following some ideas from [4].

Step 1. Transform the critical point (x^*, p^*) into the origin using the coordinate change $y = x - x^*$, $\gamma = p - p^*$. Then (1) becomes

$$\dot{y} = f(y + x^*, \gamma + p^*) \equiv g(y, \gamma).$$

Step 2. Consider the suspended dynamical system

$$\begin{aligned} \dot{y} &= g(y,\gamma), \;\; y \in R^n, \;\; \gamma \in R^1 \\ \dot{\gamma} &= 0 \end{aligned}$$

and find a Taylor approximation up to the 3rd order about $(0,0)$

$$\begin{aligned} \dot{y} &= D_y g(0,0) y + R(y,\gamma) \\ \dot{\gamma} &= 0, \end{aligned} \tag{6}$$

where $R(y,\gamma) = g_0(y,\gamma) + g^{(2)}(y,\gamma) + g^{(3)}(y,\gamma)$; $g_0(y,\gamma)$ contains all terms of $g(y,\gamma)$ depending (even linearly) on γ, and $g^{(j)}(y,\gamma)$ represents all terms in the Taylor expansion of $g(y,\gamma)$ of order j in y.

Step 3. Construct the transformation matrix T such that

$$T^{-1} D_y g(0,0) T = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$$

where the $(n-1) \times (n-1)$-matrix B has eigenvalues with nonzero real parts. Make the coordinate change $u = T^{-1} y$ to obtain

$$\dot{u} = T^{-1} D_y g(0,0) T u + T^{-1} R(Tu, \gamma).$$

Denote $G(u,\gamma) = T^{-1} R(Tu,\gamma)$, $\xi = u_1$, $\varphi = G_1(u,\gamma)$, $v = (u_2, u_3, \ldots, u_n)^{\mathrm{T}}$, $F = (G_2, G_3, \ldots, G_n)^{\mathrm{T}}$. Then (6) is topologically equivalent with

$$\begin{aligned} \dot{\xi} &= \varphi(\xi, v, \gamma) \\ \dot{v} &= Bv + F(\xi, v, \gamma) \\ \dot{\gamma} &= 0. \end{aligned} \tag{7}$$

Step 4. Reduction on the center manifold.

The center manifold $W^c(0,0,0)$ is locally represented as

$$W^c(0,0,0) = \{(\xi, v, \gamma) : \; v = V(\xi,\gamma), V(0,0) = D_\xi V(0,0) = 0\}$$

for sufficiently small $|\xi|$ and $|\gamma|$. Using the invariance of $V(\xi,\gamma)$ under the dynamics (7) we obtain that the function $V(\xi,\gamma)$ satisfies the equation

$$\begin{pmatrix} \dfrac{\partial V_1}{\partial \xi}(\xi,\gamma) \\ \vdots \\ \dfrac{\partial V_{n-1}}{\partial \xi}(\xi,\gamma) \end{pmatrix} \varphi(\xi, V(\xi,\gamma),\gamma) = B\cdot V(\xi,\gamma) + F(\xi, V(\xi,\gamma),\gamma). \tag{8}$$

Find an approximation of $V=(V_1,\dots,V_{n-1})^{\mathrm{T}}$ in the form $V_i(\xi,\gamma)=a_i\xi^2+b_i\xi\gamma+c_i\gamma^2$ with unknown coefficients a_i, b_i, c_i, $i=1,2,\dots,n-1$. The latter are determined by substituting $V(\xi,\gamma)$ in (8) and equating the coefficients of the powers ξ^2, $\xi\gamma$ and γ^2. Then the reduced dynamics on the center manifold is

$$\begin{aligned} \dot{\xi} &= \psi(\xi,\gamma), \quad \psi(\xi,\gamma)=\varphi(\xi,V(\xi,\gamma),\gamma) \\ \dot{\gamma} &= 0, \qquad\quad \xi,\gamma\in R^1. \end{aligned} \tag{9}$$

Step 5. Normal forms of the bifurcations.

Note that the equalities $\psi(0,0)=0$ and $\dfrac{\partial\psi}{\partial\xi}(0,0)=0$ (for nonhyperbolicity of $(0,0)$) are satisfied in (9). Check the genericity conditions for the fixed point bifurcations.

(i) Saddle-node bifurcation: $\dfrac{\partial\psi}{\partial\gamma}(0,0)\neq 0$ and $\dfrac{\partial^2\psi}{\partial\xi^2}(0,0)\neq 0$. The normal form of the bifurcation is

$$\dot{\xi}=\sigma_1\gamma+\sigma_2\xi^2, \quad \sigma_1=sign\,\frac{\partial\psi}{\partial\gamma}(0,0)=\pm1, \quad \sigma_2=sign\,\frac{\partial^2\psi}{\partial\xi^2}(0,0)=\pm1.$$

Figure 1 presents the saddle-node bifurcation diagrams; the stable branches are denoted by solid lines, the unstable ones by dashed lines.

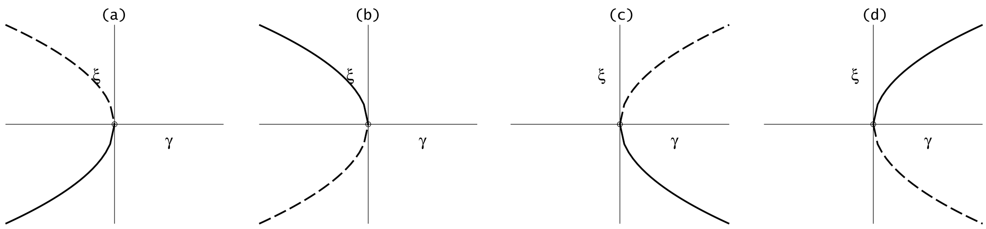

Fig. 1. Diagrams for saddle-node bifurcation; (a) $\sigma_1=\sigma_2=1$; (b) $\sigma_1=\sigma_2=-1$; (c) $-\sigma_1=\sigma_2=1$; (d) $\sigma_1=-\sigma_2=1$

(ii) Transcritical bifurcation: $\dfrac{\partial\psi}{\partial\gamma}(0,0)=0$, $\dfrac{\partial^2\psi}{\partial\xi\partial\gamma}(0,0)\neq 0$, $\dfrac{\partial^2\psi}{\partial\xi^2}(0,0)\neq 0$. The normal form of the bifurcation is

$$\dot{\xi}=\sigma_1\gamma\xi+\sigma_2\xi^2,\quad \sigma_1=sign\,\frac{\partial^2\psi}{\partial\xi\partial\gamma}(0,0)=\pm1,\quad \sigma_2=sign\,\frac{\partial^2\psi}{\partial\xi^2}(0,0)=\pm1.$$

Figure 2 presents the bifurcation diagrams; the stable branches are denoted by solid lines, the unstable ones by dashed lines.

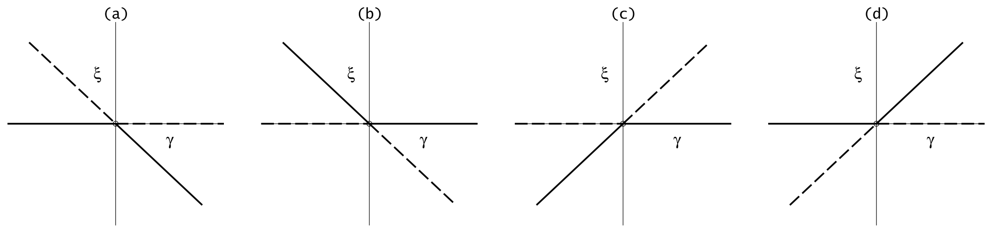

Fig. 2. Diagrams for transcritical bifurcation; (a) $\sigma_1=\sigma_2=1$; (b) $\sigma_1=\sigma_2=-1$; (c) $-\sigma_1=\sigma_2=1$; (d) $\sigma_1=-\sigma_2=1$

(iii) Pitchfork bifurcation: $\dfrac{\partial\psi}{\partial\gamma}(0,0)=0$, $\dfrac{\partial^2\psi}{\partial\xi^2}(0,0)=0$, $\dfrac{\partial^2\psi}{\partial\xi\partial\gamma}(0,0)\neq 0$ and $\dfrac{\partial^3\psi}{\partial\xi^3}(0,0)\neq 0$. The normal form of the bifurcation is

$$\dot{\xi}=\sigma_1\gamma\xi+\sigma_2\xi^3,\quad \sigma_1=sign\,\frac{\partial^2\psi}{\partial\xi\partial\gamma}(0,0)=\pm1,\quad \sigma_2=sign\,\frac{\partial^3\psi}{\partial\xi^3}(0,0)=\pm1.$$

Figure 3 presents the bifurcation diagrams; the stable branches are denoted by solid lines, the unstable ones by dashed lines.

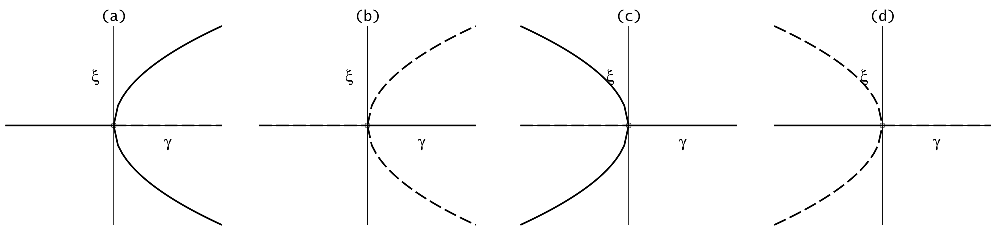

Fig. 3. Diagrams for pitchfork bifurcation; (a) $\sigma_1=-\sigma_2=1$; (b) $-\sigma_1=\sigma_2=1$; (c) $\sigma_1=\sigma_2=-1$; (d) $\sigma_1=\sigma_2=1$

3. Andronov–Hopf bifurcation. Assume that the linearization $D(x^*,p)$ of the dynamical system (1) has at (x^*,p) for sufficiently small $|p-p^*|$ a pair of complex conjugate eigenvalues $\lambda_R(p) \pm \lambda_I(p)$, such that $\lambda_R(p^*) = 0$, and all the remaining eigenvalues have nonzero real parts; denote for simplicity $\lambda_I(p^*) = \omega > 0$. In this case the dynamics (1) is topologically equivalent to a two-dimensional dynamical system, exhibiting under some genericity conditions the Andronov-Hopf bifurcation. The algorithm presented below follows [2].

Step 1. Transform the critical point (x^*,p^*) into the origin using the coordinate change $y = x - x^*$, $\alpha = p - p^*$. Then (1) becomes

$$\dot{y} = f(y + x^*, \alpha + p^*) \equiv g(y,\alpha), \quad y \in R^n, \quad \alpha \in R^1.$$

Step 2. Find a Taylor approximation of $g(y,\alpha)$ about $(0,0)$

$$\dot{y} = D_y g(0,0)y + R(y), \tag{10}$$

where the nonlinear part $R(y) = R(y,0)$ is evaluated at $\alpha = 0$ and $R(0) = D_y R(0) = 0$ holds true.

Step 3. Construct the transformation matrix T such that

$$T^{-1} D_y g(0,0) T = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & B \end{pmatrix},$$

where the $(n-2) \times (n-2)$ matrix B has eigenvalues with nonzero real parts. Make the coordinate change $u = T^{-1}y$ to obtain

$$\dot{u} = T^{-1} D_y g(0,0) T u + T^{-1} R(Tu).$$

Denote $G(u) = T^{-1}R(Tu)$. Then (10) is topologically equivalent near the origin with

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} G_1(u) \\ G_2(u) \end{pmatrix} \tag{11}$$

$$\begin{pmatrix} \dot{u}_3 \\ \vdots \\ \dot{u}_n \end{pmatrix} = B \begin{pmatrix} u_3 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} G_3(u) \\ \vdots \\ G_n(u) \end{pmatrix}. \tag{12}$$

Introduce complex coordinates $z = u_1 + iu_2$, $\bar{z} = u_1 - iu_2$, and denote for simplicity $F = G_1 + iG_2$, $H = (G_3, \dots, G_n)^{\mathrm{T}}$, $v = (u_3, \dots, u_n)^{\mathrm{T}}$. Then the system (11–12) takes the form

$$
\begin{aligned}
\dot{z} &= i\omega z + F(z, \bar{z}, v), \qquad F(0,0,0) = DF(0,0,0) = 0 \\
\dot{v} &= Bv + H(z, \bar{z}, v), \qquad H(0,0,0) = DH(0,0,0) = 0.
\end{aligned} \tag{13}
$$

Find Taylor approximations of $F(z, \bar{z}, v)$ and $H(z, \bar{z}, v)$ about $(0,0,0)$:

$$
\begin{aligned}
F(z, \bar{z}, v) &= \frac{1}{2}F_{20}z^2 + F_{11}z\bar{z} + \frac{1}{2}F_{02}\bar{z}^2 + \frac{1}{2}F_{21}z^2\bar{z} + \langle F_{10}, v\rangle z + \langle F_{01}, v\rangle \\
H(z, \bar{z}, v) &= \frac{1}{2}H_{20}z^2 + H_{11}z\bar{z} + \frac{1}{2}H_{02}\bar{z}^2,
\end{aligned} \tag{14}
$$

where $\langle \cdot, \cdot \rangle$ means the scalar product in C^{n-2} and

$$
\begin{aligned}
F_{ij} &= \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} F(z, \bar{z}, v)\bigg|_{(0,0,0)}, \quad i + j \geq 2 \\
\overline{F}_{10,i} &= \frac{\partial^2}{\partial v_i \partial z} F(z, \bar{z}, v)\bigg|_{(0,0,0)}, \quad i = 1, 2, \dots, n-2 \\
\overline{F}_{01,i} &= \frac{\partial^2}{\partial v_i \partial \bar{z}} F(z, \bar{z}, v)\bigg|_{(0,0,0)}, \quad i = 1, 2, \dots, n-2; \\
H_{ij} &= \frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j} H(z, \bar{z}, v)\bigg|_{(0,0,0)}, \quad i + j = 2.
\end{aligned}
$$

Step 4. Reduction on the center manifold.

The center manifold $W^c(0,0,0)$ is locally represented as

$$
W^c(0,0,0) = \{(z, \bar{z}, v) : \; v = V(z, \bar{z}), V(0,0) = DV(0,0) = 0\}
$$

for sufficiently small $|z|$. Denote $v = (v_1, \dots, v_{n-2})^{\mathrm{T}}$, $V = (V_1, \dots, V_{n-2})^{\mathrm{T}}$. Find an approximation of $V(z, \bar{z})$ in the form $V(z, \bar{z}) = \dfrac{1}{2}az^2 + bz\bar{z} + \dfrac{1}{2}c\bar{z}^2$, where a, b, c are unknown vectors, $b \in R^{n-2}$, a, $c \in C^{n-2}$ with $a = \bar{c}$.

Using the invariance of $v = V(z, \bar{z})$ under the dynamics (13) we obtain that the function $V(z, \bar{z})$ satisfies the equation

$$
\begin{aligned}
& ai\omega z^2 - ci\omega \bar{z}^2 + (az + b\bar{z})F(z, \bar{z}, V(z, \bar{z})) + (bz + c\bar{z})\bar{F}(z, \bar{z}, V(z, \bar{z})) \\
&= BV(z, \bar{z}) + \frac{1}{2}H_{20}z^2 + H_{11}z\bar{z} + \frac{1}{2}H_{02}\bar{z}^2.
\end{aligned} \tag{15}
$$

Equating the coefficients of z^2 and $z\bar{z}$ in (15) implies

$$\begin{aligned} z^2: &\quad (2i\omega E - B)a = H_{20} \\ z\bar{z}: &\quad -Bb = H_{01}. \end{aligned}$$

E denotes the $(n-2)\times(n-2)$ identity matrix. Solving the above linear systems delivers the coefficient vectors a, b and $c = \bar{a}$. Then (13) and (14) lead to

$$\dot{z} = i\omega z + \frac{1}{2}F_{20}z^2 + F_{11}z\bar{z} + \frac{1}{2}F_{02}\bar{z}^2 + \left(\frac{1}{2}F_{21} + \langle F_{10}, b\rangle + \langle F_{01}, \frac{1}{2}a\rangle\right) z^2\bar{z} + \cdots$$

Denote for convenience $g_{20} = F_{20}$, $g_{11} = F_{11}$, $g_{21} = F_{21} + 2\langle F_{10}, b\rangle + \langle F_{01}, a\rangle$; the reduced equation on the center manifold is

$$\dot{z} = i\omega z + \frac{1}{2}g_{20}z^2 + g_{11}z\bar{z} + \frac{1}{2}g_{21}z^2\bar{z}. \tag{16}$$

Step 5. Normal form of the Andronov-Hopf bifurcation.

Compute $\left.\dfrac{d}{d\alpha}\lambda_R(\alpha)\right|_{\alpha=0} = \lambda_R'(0)$, and the first Lyapunov coefficient $l_1(0)$ using (16),

$$l_1(0) = \frac{1}{2\omega^2} Re\left(i g_{20} g_{11} + \omega g_{21}\right).$$

If the genericity conditions $\lambda_R'(0) \neq 0$ and $l_1(0) \neq 0$ are satisfied, denote

$$\sigma_0 = \lambda_R'(0) = \pm 1, \quad \sigma_1 = \operatorname{sign} l_1(0) = \pm 1.$$

The topological normal form of the generic Andronov-Hopf bifurcation is

$$\dot{z} = (\sigma_0\alpha + i)z + \sigma_1 z|z|^2$$

or with $z = \xi + i\eta$,

$$\begin{aligned} \dot{\xi} &= \sigma_0\alpha\xi - \eta + \sigma_1(\xi^2+\eta^2)\xi \\ \dot{\eta} &= \xi + \sigma_0\alpha\eta + \sigma_1(\xi^2+\eta^2)\eta. \end{aligned} \tag{17}$$

Stability analysis of the normal form (17)

(i) $\sigma_0 = \sigma_1 = 1$: $(0,0)$ is asymptotically stable for $\alpha < 0$ and unstable for $\alpha > 0$; an unstable limit cycle exists for $\alpha < 0$ (subcritical bifurcation), Figure 4. (ii) $\sigma_0 = \sigma_1 = -1$: $(0,0)$ is asymptotically stable for $\alpha > 0$ and unstable for

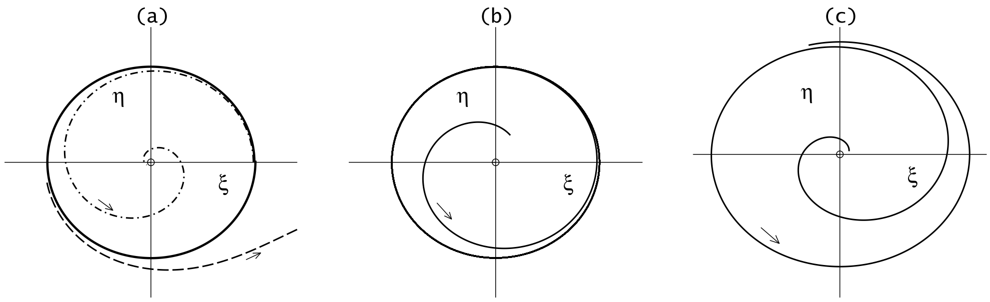

Fig. 4. Subcritical Andronov-Hopf bifurcation in the case $\sigma_0 = \sigma_1 = 1$; (a) $\alpha < 0$; (b) $\alpha = 0$; (c) $\alpha > 0$

$\alpha < 0$; a stable limit cycle exists for $\alpha < 0$ (supercritical bifurcation).

(iii) $\sigma_0 = +1$, $\sigma_1 = -1$: $(0, 0)$ is asymptotically stable for $\alpha < 0$ and unstable for $\alpha > 0$; a stable limit cycle exists for $\alpha > 0$ (supercritical bifurcation), Figure 5.

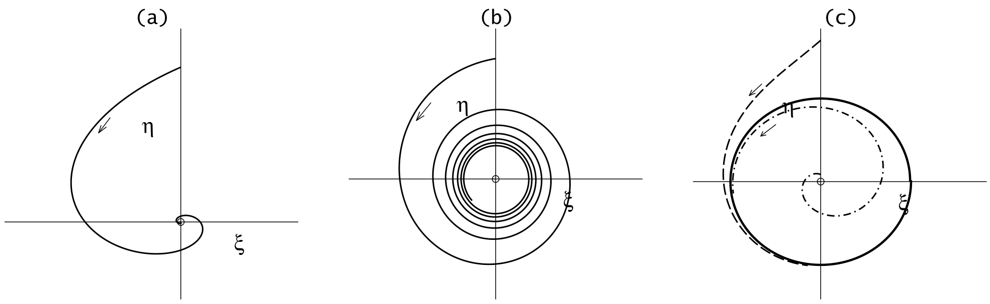

Fig. 5. Supercritical Andronov-Hopf bifurcation in the case $\sigma_0 = +1$, $\sigma_1 = -1$; (a) $\alpha < 0$; (b) $\alpha = 0$; (c) $\alpha > 0$

(iv) $\sigma_0 = -1$, $\sigma_1 = +1$: $(0, 0)$ is asymptotically stable for $\alpha > 0$ and unstable for $\alpha < 0$; an unstable limit cycle exists for $\alpha > 0$ (subcritical bifurcation).

4. Implementation in Maple and examples. The two algorithms presented above are implemented in CAS *Maple 13*© and designed as a Maple package BifTools. The procedure for symbolic calculation of bifurcations of the steady states with a single zero eigenvalue of the Jacobian is calcOneZeroEigenvalueBif; the procedure for symbolic calculation of Andronov-Hopf bifurcation of the equilibrium points is calcHopfBif. Both procedures require as Maple input the ODEs system, the phase variables, the bifurcation parameter, the critical steady state and the parameter bifurcation value. Below we present two examples to demonstrate the facilities of the package.

Example 1 [5]. After entering and executing the commands

```
> with(BifTools):
> ode1:=diff(x(t),t)=x(t)*(1-x(t))+a*x(t)*y(t)/(x(t)+y(t))-h:
  ode2:=diff(y(t),t)=y(t)*(-d+b*x(t)/(x(t)+y(t))):
> BifTools:-calcOneZeroEigenvalueBif([ode1,ode2],[x(t),y(t)],
          h,[1/2,0],1/4);
```

the following results are displayed

– The ODEs system –

$$\begin{aligned} \frac{d}{dt}x(t) &= x(t)(1-x(t)) + \frac{ax(t)y(t)}{x(t)+y(t)} - h \\ \frac{d}{dt}y(t) &= y(t)\left(-d + \frac{bx(t)}{x(t)+y(t)}\right) \end{aligned}$$

– The bifurcation point –

$$[x(t), y(t)] = \left[\frac{1}{2}, 0\right], \quad h = \frac{1}{4}$$

– Results from the analysis –

"The reduced ODEs system":

$$\begin{aligned} \frac{d}{dt}\_X(t) &= \frac{(-d+b)\gamma}{a} + \frac{a\,\_X(t)^2}{-d+b} \\ \frac{d}{dt}\gamma &= 0 \end{aligned}$$

"Saddle-node bifurcation of the equilibrium point"

"Eigenvalues of the Jacobian at the equilibrium point":

$$[0, -d+b]$$

Example 2 [2]. Entering and executing the commands

```
> with(BifTools);
> ode1:=diff(x1(t),t)=r*x1(t)*(a+x1(t))*(1-x1(t))-c*x1(t)*x2(t):
  ode2:=diff(x2(t),t)=-a*d*x2(t)+(c-d)*x1(t)*x2(t):
> assume(c>0): assume(d>0): assume(r>0):
> c:=2*d:  a0:=(c-d)/(c+d):
  x0[1]:=d/(c+d):  x0[2]:=(r*c)/(c+d):
> BifTools:-calcHopfBif([ode1,ode2],[x1(t),x2(t)],a,[x0[1],x0[2]],a0);
```

produce the following results

– The ODEs system –

$$\begin{aligned}\frac{d}{dt}x1(t) &= r\tilde{}\,x1(t)(a+x1(t))(1-x1(t)) - c\tilde{}\,x1(t)x2(t)\\ \frac{d}{dt}x2(t) &= -ad\tilde{}\,x2(t) + (c\tilde{} - d\tilde{})x1(t)x2(t)\end{aligned}$$

– The bifurcation point –

$$[x1(t), x2(t)] = \left[\frac{1}{3}, \frac{2}{9}\frac{r\tilde{}}{d\tilde{}}\right], \quad a = \frac{1}{3}$$

– Results from the analysis –

"The transversality condition":

$$\left.\frac{d}{d\alpha}\lambda_R(\alpha)\right|_{\alpha=0} = \frac{1}{6}r\tilde{} - \frac{1}{2}d\tilde{}$$

"The first Lyapunov coefficient":

$$l_1(0) = -\frac{9}{2}\frac{\sqrt{d\tilde{}}\sqrt{3}}{\sqrt{r\tilde{}}}$$

"The normal form":

$$\frac{d}{dt}\_X(t) = \sigma_0\alpha\,\_X(t) - \_Y(t) - (\_X(t)^2 + \_Y(t)^2)\,\_X(t)$$

$$\frac{d}{dt}\_Y(t) = \_X(t) + \sigma_0\alpha\,\_Y(t) - (\_X(t)^2 + \_Y(t)^2)\,\_Y(t)$$

$$\sigma_0 = sign\left(\frac{1}{6}\,r^{\sim} - \frac{1}{2}\,d^{\sim}\right)$$

5. Conclusion and future work. The paper is devoted to algorithms for one-parameter local bifurcations of equilibrium points of dynamical systems and their implementation in the computer algebra system *Maple 13*©. The designed package BifTools consists of two main procedures: calcOneZeroEigenvalueBif for symbolic calculation of bifurcations of the steady states with a single zero eigenvalue of the Jacobian, and calcHopfBif for symbolic calculation of Andronov-Hopf bifurcation of the equilibrium points. As mentioned above, both procedures require as input the critical steady state and the parameter bifurcation value, which means that they should be known to the user in advance. It is natural to extend the package by additional procedures for (symbolic and/or numeric) steady states computations, determining thereby critical parameter values of possible bifurcations of the equilibrium points. Such an extension is already in preparation.

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