

**QUADRATIC TIME COMPUTABLE INSTANCES OF
MAXMIN AND MINMAX AREA TRIANGULATIONS OF
CONVEX POLYGONS***

Tigran Mirzoev, Tzvetalin S. Vassilev

ABSTRACT. We consider the problems of finding two optimal triangulations of a convex polygon: MaxMin area and MinMax area. These are the triangulations that maximize the area of the smallest area triangle in a triangulation, and respectively minimize the area of the largest area triangle in a triangulation, over all possible triangulations. The problem was originally solved by Klincsek by dynamic programming in cubic time [2]. Later, Keil and Vassilev devised an algorithm that runs in $O(n^2 \log n)$ time [1]. In this paper we describe new geometric findings on the structure of MaxMin and MinMax Area triangulations of convex polygons in two dimensions and their algorithmic implications. We improve the algorithm's running time to quadratic for large classes of convex polygons. We also present experimental results on MaxMin area triangulation.

ACM Computing Classification System (1998): I.3.5.

Key words: computational geometry, triangulation, convex polygon, dynamic programming.

*Preliminary version of this paper was presented at XII Encuentros de Geometría Computacional, Valladolid, Spain, June 2007.

1. Preliminaries. For a given 2-dimensional point set, a triangulation is a maximal set of non-intersecting straight line segments with vertices in the point set. All the faces of a triangulation, except for the unbounded face, are triangles, hence the name.

Triangulations of point sets in the 2-dimensional Euclidean plane have been studied in computational geometry both theoretically and in connection with their numerous applications to graphics, interpolation, and finite elements methods, to name but a few. By an optimal triangulation we mean a triangulation that optimizes a given criterion, often called quality measure, over all possible triangulations of a given point set. The optimization problem is not trivial, given the fact that even for a point set in convex position, there are exponentially many in n (the number of points) different triangulations. The optimization criterion might involve minimization or maximization that is done over elements of individual triangles, such as edge lengths, angles, area, etc., or over similar characteristics of the entire triangulation.

For a point set in general position many optimal triangulations are polynomially computable, for example the Delaunay Triangulation (MaxMin angle), MinMax angle triangulation, MinMax edge length triangulation, etc. Other optimal triangulations are either known to be NP-hard as the famous Minimum Weight Triangulation [3], or do not have polynomial solutions as the MinMax and MaxMin Area triangulations.

When the point set is in convex positions, there is a general dynamic programming algorithm by Klincsek [2] that finds the optimal triangulation in $\Theta(n^3)$ time and $\Theta(n^2)$ space with respect to any decomposable measure. A decomposable measure is a measure for which the optimal solution for a convex polygon can be obtained by combining the solutions for its subpolygons. MaxMin and MinMax area are decomposable measures. The problem of finding the MaxMin and MinMax area triangulations was studied by Keil and Vassilev [1]. They developed an algorithm that solves both problems in $O(n^2 \log n)$ time and $O(n^2)$ space based on solving only a quadratic number of subproblems and performing a logarithmic time search for the optimal triangulations over a quadratic number of possible cases. Further, the MaxMin area triangulation can be computed in $O(n^2 \log \log n)$ time by using van Emde-Boas priority queues. Here we will show that for certain classes of convex polygons it is possible to compute both MaxMin and MinMax area triangulations in quadratic time and space. To achieve this, we are going to further analyze the structure and the geometric properties of these two optimal triangulations.

First we have to introduce some notation. Most of it will be the original

notation used by Keil and Vassilev. We assume that the vertices of the convex polygon P are enumerated modulo n in clockwise order. P_{ij} is the convex subpolygon of P , containing the vertices $i, i+1, \dots, j-1, j$ in this order. The convex subpolygon P_{ji} is called a *complementary* subpolygon of P_{ij} , their union is P . One important result that we are going to use is the unimodality of the distances between the vertices of a convex polygon, in their order along the boundary, and any given edge (or diagonal) of the convex polygon. This result is due to Toussaint [4]. For every edge or diagonal ij of P , we denote by $Top(i, j)$ the vertex of P_{ij} that is farthest from ij . If there are two such vertices, we are taking the one that precedes the other in the enumeration (clockwise order). The term *zonality* of a subpolygon is related to the sum of its two base angles. Precisely, the zonality $z(i, j)$ of P_{ij} is defined as

$$z(i, j) = \left\lceil \frac{2(\angle ji(i+1) + \angle (j-1)ji)}{\pi} \right\rceil$$

With respect to this definition, for each vertex i of P we define $MaxCW(i)$ and $MaxCCW(i)$. $MaxCW(i)$ is the last vertex k along the boundary, starting from i and going clockwise, such that the subpolygon P_{ik} has zonality of 2 or less. Similarly, $MaxCCW(i)$ is the last vertex m along the boundary, starting from i and going counterclockwise, such that the subpolygon P_{mi} has zonality of 2 or less. It is shown in [1] that $MaxCCW(i) = Top(i-1, i)$ and $MaxCW(i) = Top(i, i+1)$ or $Top(i, i+1) + 1$.

2. Structural properties of the optimal triangulations. Keil and Vassilev's algorithm is based on the fact that the optimal triangulations, MaxMin and MinMax area, both contain either a 2-2 diagonal (i. e., a diagonal ij such that $z(i, j) \leq 2, z(j, i) \leq 2$) or a 2-2-2 triangle (i. e., triangle ijk such that $z(i, j) \leq 2, z(j, k) \leq 2, z(k, i) \leq 2$).

We are going to show that there are at most a linear number of 2-2 diagonals. Further we are going to study the 2-2-2 triangles, and outline conditions under which they can be removed, obtaining better triangulations. We will show that in such cases there is at most a linear number of *significant* (definition to follow) degenerated 2-2-2 triangles. We are going to show examples of point sets with a cubic number of significant non-degenerated triangles. Finally, we will discuss the optimal area triangulations of regular polygons.

We start with the 2-2 diagonals.

Observation 2.1. *Let $(i-1)i$ be a boundary edge of P and $j = Top(i-$*

$1, i$). Then $(i-1)j$ and ij are 2-2 diagonals. Further, if $j(j+1) \parallel (i-1)i$, then $(i-1)(j+1)$ and $i(j+1)$ are also 2-2 diagonals.

Proof. The stated properties follow directly from the definition of Top . If we construct a line through j parallel to the edge $(i-1)i$, then the entire polygon P lies inside this strip and thus $z(i-1, j) \leq 2, z(i, j) \leq 2, z(j, i-1) \leq 2, z(j, i) \leq 2$. Therefore $(i-1)j$ and ij are 2-2 diagonals. A similar argument applies to $(i-1)(j+1)$ and $i(j+1)$ when $j(j+1) \parallel (i-1)i$. Note that when $j(j+1) \not\parallel (i-1)i$, we have $z(i-1, j+1) > 2$ and also $z(i, j+1)$ may be greater than 2, although this is not necessarily the case. \square

This lemma shows that for every boundary edge the diagonals that connect it to its Top vertex are 2-2 diagonals. Further, we are going to show that every 2-2 diagonal is of this type.

Observation 2.2. *Let ij be a 2-2 diagonal of P . Then at least one of the vertices i and j is a Top vertex for one of the boundary edges adjacent to the other.*

Proof. We are going to assume the contrary. Namely, ij is a 2-2 diagonal of P such that $Top(i-1, i) \neq j, Top(i, i+1) \neq j, Top(j-1, j) \neq i, Top(j, j+1) \neq i$. Consider the four angles adjacent to the diagonal ij . Without loss of generality we can assume that the smallest of them is $\angle ji(i+1) = \alpha$. In order for j not to be $Top(i+1, i)$ we should have $\angle ij(j+1) > \alpha$. If $\angle ij(j+1) = \alpha$, then $i(i+1) \parallel j(j+1)$ and therefore $j = Top(i, i+1), i = Top(j, j+1)$, which contradicts our assumption. Since $\angle ij(j+1) > \alpha$, and since ij is a 2-2 diagonal, we have $\angle (i-1)ij < \pi - \angle ij(j+1)$. Therefore both vertices $(i-1)$ and $(i+1)$ lie strictly inside the strip determined by the line supporting the edge $j(j+1)$ and the line parallel to $j(j+1)$ through the vertex i . This, according to the definition, means that $i = Top(j, j+1)$, which contradicts our assumption. The claim is established. \square

This lemma shows that every 2-2 diagonal can be assigned to an edge incident to one of its endpoints, such that the other endpoint is Top for this edge. Combined with Observation 2.1 which showed that there are at most two such 2-2 diagonals, we can establish the following result.

Lemma 2.3. *The number of 2-2 diagonals in a convex polygon P with n vertices is at most $2n$.*

This bound is tight. Consider for example, the regular octagon. It has exactly sixteen different 2-2 diagonals. Thus we have proven that although there are $O(n^2)$ diagonals in a convex polygon of n vertices, those of them that are

2-2 diagonals are at most $O(n)$. Therefore, after the dynamic programming is performed, as per the algorithm of Keil and Vassilev, only linear time is sufficient to check all possible triangulations that have a 2-2 diagonal and determine the best of them.

Next we are going to focus on the 2-2-2 triangles. Recall that for a triangle that has no boundary edge (so-called *internal triangle*) we usually improve the triangulation by flips. A *flip* is a retriangulation in which one of the diagonals of a convex quadrilateral is replaced by the other diagonal. Also, recall that a flip improves both area quality measures, MaxMin and MinMax area, simultaneously.

Definition 2.4. *A 2-2-2 triangle in a triangulation is called significant if there is no flip retriangulation that involves this triangle and gives a triangulation with better quality. All other 2-2-2 triangles are called insignificant.*

In order to handle the insignificant 2-2-2 triangles, we are going to introduce the following lemma.

Lemma 2.5. *Consider a 2-2-2 triangle Δpqr . If any of the 2-zone subpolygons P_{pq}, P_{qr} , or P_{rp} fits entirely into the parallelogram formed by the initial triangle Δpqr and the triangle congruent to it and incident to its sides pq , qr , and rp respectively, outside of Δpqr , then Δpqr is insignificant. Refer to Figure 1 for an illustration.*

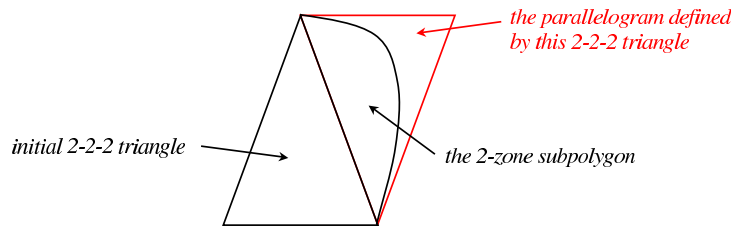


Fig. 1. The parallelogram fitting

Proof. To prove the lemma, we are going to retriangulate in a way that improves the quality of the triangulation. Without loss of generality, we can assume that the subpolygon P_{qr} fits entirely into the parallelogram formed by the initial 2-2-2 triangle Δpqr and the triangle $\Delta qp'r$ congruent to it. This situation is illustrated in Figure 2. If the diagonal qr is connected to the vertex s in the optimal triangulation of P_{qr} , we can replace (flip) the diagonal qr with the diagonal ps . The resulting triangulation has equal or better quality because $A_{\Delta pqr} > A_{\Delta psr}$ and $A_{\Delta pqr} > A_{\Delta pqs}$. Thus the largest area triangle in the former triangulation, Δpqr , has been replaced by triangles with smaller areas.

Similarly, the area of the triangle $\triangle qsr$ has been increased, since $A_{\triangle psr} > A_{\triangle qsr}$ and $A_{\triangle pqs} > A_{\triangle qsr}$. Note, that the triangle $\triangle qsr$ may not have been the smallest area triangle in the initial triangulation. Thus the precise conclusion is that we have obtained a retriangulation which is strictly better in terms of MinMax area, and no worse in terms of MaxMin area. Continuing the retriangulation in the above manner inside $\angle spq$ and $\angle rps$, we obtain a fan triangulation from vertex p . \square

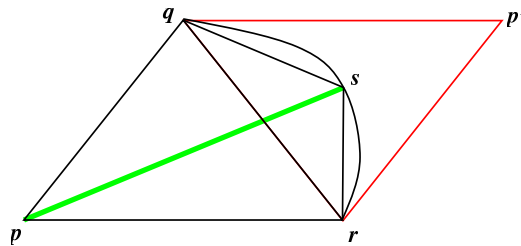


Fig. 2. The retriangulation inside the parallelogram

Equipped with the result of Lemma 2.5, we are going to discuss when it is possible to ensure that at least one of the three subpolygons associated with a 2-2-2 triangle satisfies the premises of Lemma 2.5.

Let $\triangle pqr$ be a 2-2-2 triangle in a triangulation of P . Refer to Figure 3 for the construction that follows. We denote by p', q', r' the vertices of the triangles congruent to $\triangle pqr$ and incident to its sides qr, rp, pq , respectively. It is easy to see that $pq \parallel p'q', qr \parallel q'r', rp \parallel r'p'$ and the triangle $\triangle p'q'r'$ outscribes our initial triangle $\triangle pqr$. Lines $p'q', q'r', p'r'$ are drawn in red in Figure 3.

Naturally, if any of the subpolygons P_{pq}, P_{qr}, P_{rp} lie entirely inside the triangles $\triangle pr'q, \triangle qp'r, \triangle rq'p$, we will apply Lemma 2.5 and remove $\triangle pqr$ from the triangulation. When will this be possible? To answer this question, consider the three lines t_p, t_q, t_r that are tangent to P at its vertices p, q, r , respectively. These lines are drawn in blue in Figure 3. Note that these lines are not unique, i. e., we have some freedom to choose them as is convenient to us. These three lines form a triangle, $\triangle s_{pq}s_{qr}s_{rp}$ (the case when two of them are parallel will be considered later). Without loss of generality consider the position of the lines t_q and t_r with respect to the triangle $\triangle qp'r$. There are three possibilities:

1. both lines intersect the interior of the triangle or coincide with its side(s)
2. neither line intersects the interior of the triangle
3. exactly one of the lines intersects the interior of the triangle or coincides with its side

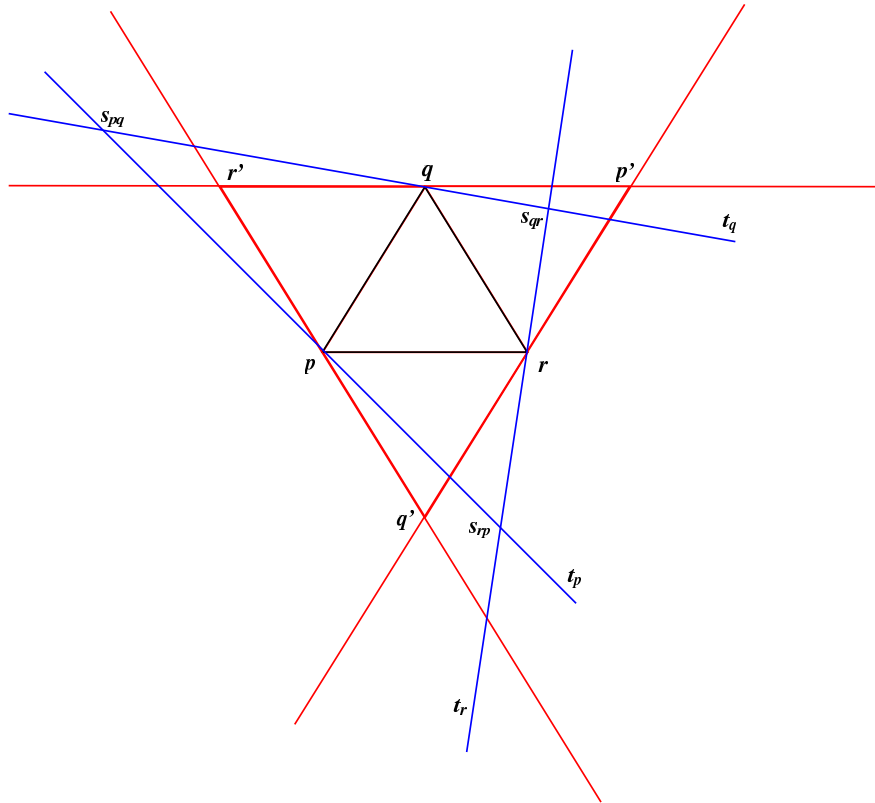


Fig. 3. Outscribing a 2-2-2 triangle

In case 1, s_{qr} lies inside $\triangle qp'r$ (here we include the case when the point s_{qr} coincides with p'). Therefore, the entire subpolygon P_{qr} lies inside $\triangle qp'r$ (or at most coincides with it). Therefore, $\triangle pqr$ is insignificant and can be removed from the triangulation by the method of Lemma 2.5.

In case 2, we are going to show that one of the other two points s_{pq}, s_{rp} lies inside the respective red triangle $\triangle pr'q, \triangle rq'p$. To see that this is true, consider the fact that t_q not intersecting the interior of $\triangle qp'r$ means t_q intersects the interior of $\triangle pr'q$. Similarly t_r does not intersect the interior of $\triangle qp'r$, therefore t_r intersects the interior of $\triangle rq'p$. Now consider the position of the line t_p . It intersects the interior of exactly one of the two triangles $\triangle pr'q, \triangle rq'p$ or coincides with the line $q'r'$. Whichever the case, we have a pair of lines that satisfy the description of case 1 that was considered above. Thus the triangle $\triangle pqr$ is insignificant and will be removed from the triangulation through an application

of Lemma 2.5. Note that the case when $t_q \parallel t_r$ belongs here. If t_q and t_r are parallel, neither of them can intersect the interior of $\Delta qp'r$, because it will mean that the other will intersect the interior of Δpqr , which is impossible (all the lines t_p, t_q, t_r by construction lie outside the entire polygon P and are only tangent to it in the points p, q, r , respectively).

Case 3 is our “bad” case. We cannot guarantee a retriangulation of the polygon that removes Δpqr from the triangulation. Moreover it is easy to construct a six-point example in which such 2-2-2 triangle will be part of both MaxMin and MinMax area triangulation. Thus case 3 corresponds to the case when the triangle Δpqr is significant.

3. Polygons with quadratic time computability of MaxMin and MinMax area triangulations. In their considerations of the structure of the optimal area triangulations, Keil and Vassilev showed that the optimal triangulations, both MaxMin and MinMax, contain either a 2-2 diagonal (in which case there were no internal triangles at all) or at most one internal 2-2-2 triangle. It is clear from the construction in the preceding chapter that if we have a triangulation with an insignificant 2-2-2 triangle, this triangulation is not going to be the optimal one, since it can be replaced by a better triangulation that has no non-degenerate 2-2-2 triangles (it may have 2-2-2 triangles with boundary edges). From these considerations we can deduct our main result:

Theorem 3.1. *Let P be a convex polygon with n vertices in the plane. If there are at most $O(n^2)$ possible significant 2-2-2 triangles formed by the vertices of P , then the MaxMin and MinMax area triangulations of P can be computed in $O(n^2)$ time and space.*

Proof. We recall from [1] that by dynamic programming we can compute the optimal triangulations of all up to 2-zone subpolygons of P in $O(n^2)$ time. In order to find the optimal triangulation, based on the knowledge of its structure, we have to check:

- All possible triangulations containing a 2-2 diagonal:
Based on our results here, Lemma 2.3, there are $O(n)$ such diagonals; checking each of them requires two lookups in the dynamic programming table, i. e., constant time, therefore all 2-2 diagonals are checked in $O(n)$ time.
- All possible triangulations containing a 2-2-2 triangle with a boundary edge:
There are $O(n^2)$ different triangles with a boundary edge. Therefore there are at most $O(n^2)$ of them that are 2-2-2 triangles. Checking each of them

requires constant time (two lookups in the dynamic programming table for the associated subproblems), and thus they are all checked in $O(n^2)$ time.

- All possible triangulations containing a significant 2-2-2 triangle:
 There are at most $O(n^2)$ significant triangles according to our assumption. Each of them is checked in constant time, three lookups in the dynamic programming table for the three associated subproblems. Therefore all significant 2-2-2 triangles are checked in total $O(n^2)$ time. \square

It is an open problem to better describe the class of all convex polygons that have at most a quadratic number of possible significant 2-2-2 triangles. It is also not straightforward to describe the class of convex polygons that have no significant 2-2-2 triangles at all. However, we are going to show that the polygons that can be inscribed in a circle (i. e., polygons whose vertices lie on a given circle) are a subclass of this class.

Lemma 3.2. *Let P be a polygon all the vertices of which are on a given circle C . All possible 2-2-2 triangles in P are insignificant.*

Proof. Consider a triangle $\triangle abc$ in P that is a 2-2-2 triangle. If we use the standard notation for the angles of the triangle, we can assume that $\alpha \leq \beta \leq \gamma \leq \pi/2$. This means that every possible 2-2-2 triangle is necessarily non-obtuse. An obtuse angle will have the subproblem associated with its opposite side of zonality more than 2. If we take the triangle $\triangle a'b'c'$ formed by the tangents to C at the points a, b, c , we can show that the vertex (in this case a') opposing the smallest angle of the original triangle ($\angle a = \alpha$ according to our assumption) gives us the desired retriangulation according to Lemma 2.5. Consider triangle $\triangle ca'b$. It is an isosceles triangle, $a'c = a'b$ as tangents from a' to C . The two base angles are $\angle bca' = \angle a'bc = \alpha$. Thus we have $\angle bac + \angle ca'a = \alpha + (\gamma + \alpha) = 2\alpha + \gamma \leq \alpha + \beta + \gamma = \pi$ and $\angle bac + \angle a'ba = \alpha + (\beta + \alpha) = 2\alpha + \beta \leq \alpha + \beta + \gamma = \pi$. Therefore the point a' lies inside the parallelogram formed by ba and ac , and the triangle $\triangle abc$ is insignificant. \square

This lemma does not immediately result in an improvement of the quadratic running time of the algorithm as we still face the need to compute as many as a quadratic number of subproblems' optimal solutions as well as the possibility of a quadratic number of degenerate 2-2-2 triangles that need to be considered in the search for the optimal triangulation of P . However, if P is a regular polygon, we can do better.

4. MaxMin and MinMax area triangulations of regular polygons. Let R_n be the regular polygon with n vertices.

Observation 4.1. *Every triangulation of R_n is a MaxMin area triangulation.*

Proof. Using the unimodality of the distance between an edge and the ordered chain of vertices of the polygon [4], we can see that the smallest possible area of a triangle that has a boundary edge is that of an ear triangle, i. e., a triangle that has two boundary edges. Combining this with the previously derived result about the structure of the optimal triangulation, namely that there are at least two ears in any triangulation of a convex polygon [1], we conclude that the claim is true. \square

So it turns out that maximization of the minimum area is trivial for regular polygons. The minimization of a maximum area involves the following.

Lemma 4.2. *The MinMax area triangulation of R_n always contains a boundary triangle in which the edge is connected to its Top vertex. We call such a triangle an edge-to-Top triangle.*

Proof. First, consider the case when n is odd. Because of the symmetry, we can base our considerations on the edge $n1$. For this edge $Top(n, 1) = \left\lceil \frac{n}{2} \right\rceil = \frac{n+1}{2} = j$. We are going to show that all possible 2-2 diagonals incident to the vertex j are $1j$ and jn . Since a line through j , parallel to the line supporting the edge $n1$, defines a strip entirely containing R_n , both $1j$ and jn are 2-2 diagonals. Consider now the diagonal $2j$. We have to find the sum of the angles $\angle 12j$ and $\angle 2j(j+1)$ in order to show that $2j$ is not a 2-2 diagonal. The angle $\angle 12j$ inscribes an arc that equals $\left\lceil \frac{n}{2} \right\rceil$ times the arc of a single edge, which equals $\frac{2\pi}{n}$. Thus, $\angle 12j = \frac{1}{2} \left\lceil \frac{n}{2} \right\rceil \frac{2\pi}{n} = \frac{1}{2} \frac{n+1}{2} \frac{2\pi}{n} = \frac{(n+1)\pi}{2n}$. Further, $\angle 12j = \angle 2j(j+1)$, and therefore $\angle 12j + \angle 2j(j+1) = 2\angle 12j = 2 \frac{(n+1)\pi}{2n} = \frac{(n+1)\pi}{n} > \pi$. Thus, $2j$ is not a 2-2 diagonal. Because of the symmetry, $j(n-1)$ is also not a 2-2 diagonal. All other diagonals incident to j are similarly shown not to be 2-2 diagonals. Now consider the MinMax area triangulation of R_n . It contains either a 2-2 diagonal, which leads to an edge-to-Top triangle, or an 2-2-2 triangle with a boundary edge, the only possibility of which is the same edge-to-Top triangle. Finally, the MinMax area triangulation of R_n when n is odd has one of its boundary edges connected to its Top and the rest of the triangulation consists of only boundary triangles.

Let now n be even. Most of the considerations for the odd case apply here. However, we have to have in mind that now $Top(n, 1) = \frac{n}{2}$. Note that in R_n each edge has a parallel counterpart. Again, if we denote $j = Top(n, 1)$ the diagonals $1j$ and jn are 2-2 because of the fact that the edges $n1$ and $j(j + 1)$ are parallel. Similarly, the diagonals $1(j + 1)$ and $(j + 1)n$ are 2-2 diagonals. We note that the diagonals $1(j + 1)$ and jn are actually diameters of the circle circumscribed about R_n . Because of the symmetry, here we have a third diagonal incident to j , namely $j(n - 1)$, which is 2-2 diagonal. However, as in the odd case it is still part of an edge-to- Top triangle, since j can be considered $Top(n - 1, n)$ (actually $j - 1 = Top(n - 1, n)$). Consequently, each possible 2-2 diagonal leads to an edge-to- Top triangle, and each possible 2-2-2 boundary triangle is an edge-to- Top triangle. \square

Based on these considerations, we obtain the following result for the optimal area triangulations of a regular polygon:

Theorem 4.3. *For a regular polygon R_n in the plane, both the MaxMin and the MinMax area triangulation are computable in optimal $\Theta(n)$ time and space.*

Proof. Any triangulation of R_n is MaxMin area triangulation, thus we can just take the canonical triangulation—a fan from one of the vertices. The canonical triangulation is also going to be a MinMax area triangulation as it will contain an edge-to- Top triangle. Computing the canonical triangulation requires $\Theta(n)$ time and space, since it has $\Theta(n)$ edges, and this is optimal as any triangulation of a set of n points requires at least $O(n)$ time and space. \square

5. Experimental results. The results presented in this section are part of the graduate thesis work [6] of the first author on the efficient implementation of the algorithms for MaxMin and MinMax area triangulations presented in [1]. In Figure 4 we show the MaxMin area triangulations of a 50-point and a 100-point convex polygons inscribed in a circle. The vertices were obtained by random choice of points on the circle. It is evident from the figure that the triangulations exhibit the structure claimed in this paper, namely a sequence of fans centered at the polygon's vertices without internal triangles.

The algorithms of Keil and Vassilev were implemented in their original version, which has $O(n^2 \log n)$ running time. The running times of both algorithms were measured over instances of size 20 to 400. The algorithms were run on 100 different point sets of each size, and the running times were averaged. The running times are summarized in Tables 1 and 2. For illustrative purposes,

these experimental data were compared to n^2 and $n^2 \log n$. The trends that data shows are: $T_{MaxMin}(n) = 0.3n^2$ and $T_{MinMax}(n) = 0.23n^2 \log n$. As expected, both timing functions are (low!) constant multiples of n^2 and $n^2 \log n$, respectively. Further, the computation of the MinMax area triangulation is slower than the computation of MaxMin area triangulation as discussed in the original paper [1]. These findings partially support the conjecture that in fact these algorithms have quadratic running time on most input instances.

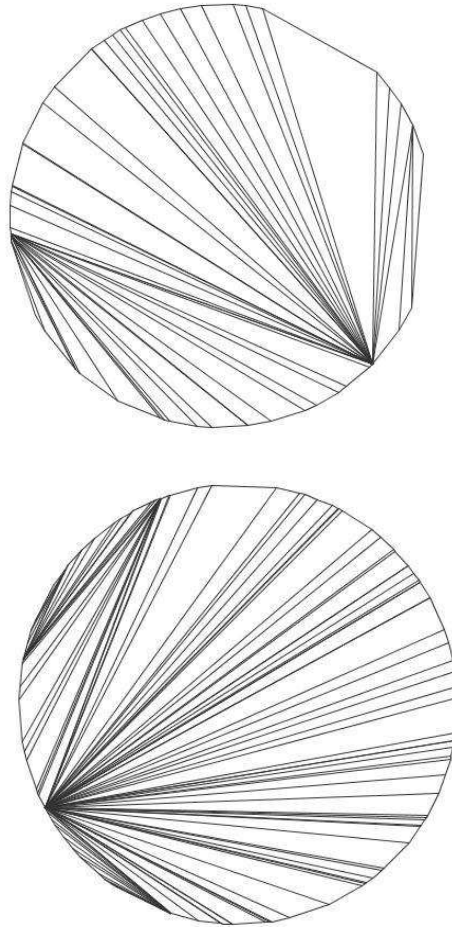


Fig. 4. MaxMin area triangulations of convex polygons inscribed in a circle

Table 1. Running times of the MaxMin area triangulation algorithm

Polygon size, n	Average time, ms	Std. deviation, ms	Min. time, ms	Max. time, ms
20	105	8	91	112
30	262	12	250	281
50	741	11	734	766
100	3038	15	3016	3063
200	12263	142	12142	12563
300	27846	347	27563	28736
400	46560	565	46002	48078

Table 2. Running times of the MinMax area triangulation algorithm

Polygon size, n	Average time, ms	Std. deviation, ms	Min. time, ms	Max. time, ms
20	239	8	231	253
30	695	20	672	735
50	2252	41	2188	2328
100	10651	98	10517	10767
200	49013	538	47785	49675
300	116553	1167	114956	118002
400	200959	1684	197515	202465

6. Conclusion. We are interested in furthering the results of this paper towards a general quadratic time algorithm for optimal area triangulations of any convex polygon. Despite the fact that we can construct examples of point sets with $\Theta(n^3)$ significant 2-2-2 triangles [5], it may be possible to establish a hierarchy among them that will allow quadratic time computation of the optimal triangulations. To our knowledge, the question of polynomial computability of MaxMin and MinMax area triangulations of a point set in general position remains open.

REFERENCES

- [1] KEIL J. M. , T. S. VASSILEV. Algorithms for Optimal Area Triangulations of Convex Polygons. *Computational Geometry: Theory and Applications*, **35** (2006), No 3, 173–187.

- [2] KLINCSEK G. T. Minimal Triangulations of Polygonal Domains. *Annals of Discrete Mathematics*, **9** (1980), 121–123.
- [3] MULZER W. , G. ROTE. Minimum Weight Triangulation is NP-hard. In: Proceedings of the 22nd Annual ACM Symposium on Computational Geometry, June 2006, 1–10.
- [4] Toussaint G. Complexity, Convexity and Unimodality. *International Journal of Computer and Information Sciences*, **13** (1984), No 3, 197–217.
- [5] Van Kreveld M. Personal Communication, March 2007.
- [6] Mirzoev T. Efficient Implementation of Optimal Area Triangulation Algorithms. Master's Thesis, North Carolina Central University, December 2007.

Tigran Mirzoev
SAS Institute Inc.
100 SAS Campus Drive, Room S5017,
Cary, NC 27513–8617 USA.
e-mail: tigran.mirzoev@sas.com

Tzvetalin S. Vassilev
Department of Computer Science and Mathematics
Nipissing University
100 College Drive, Box 5002, North Bay
Ontario P1B 8L7, Canada.
e-mail: tzvetalv@nipissingu.ca

Received February 22, 2010
Final Accepted July 23, 2010