Serdica J. Computing **4** (2010), 321–334

Serdica
Journal of Computing

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

APPROXIMATING THE MAXMIN AND MINMAX AREA TRIANGULATIONS USING ANGULAR CONSTRAINTS*

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Abstract. We consider sets of points in the two-dimensional Euclidean plane. For a planar point set in general position, i.e. no three points collinear, a triangulation is a maximal set of non-intersecting straight line segments with vertices in the given points. These segments, called edges, subdivide the convex hull of the set into triangular regions called faces or simply triangles. We study two triangulations that optimize the area of the individual triangles: MaxMin and MinMax area triangulation. MaxMin area triangulation is the triangulation that maximizes the area of the smallest area triangle in the triangulation over all possible triangulations of the given point set. Similarly, MinMax area triangulation is the one that minimizes the area of the largest area triangle over all possible triangulations of the point set. For a point set in convex position there are $O(n^2 \log n)$ time and $O(n^2)$ space algorithms that compute these two optimal area triangulations. No polynomial time algorithm is known for the general case. In this paper we present an approach

ACM Computing Classification System (1998): I.3.5.

Key words: Computational geometry, triangulation, planar point set, angle restricted triangulation, approximation, Delauney triangulation.

*A preliminary version of this paper was presented at XI Encuentros de Geometría Computacional, Santander, Spain, June 2005.

to approximation of the MaxMin and MinMax area triangulations of a general point set. The algorithm, based on angular constraints and perfect matchings between triangulations, runs in $O(n^3)$ time and $O(n^2)$ space. We determine the approximation factors as functions of the minimal angles in the optimal (unknown) triangulation and the approximating one.

1. Introduction. We consider the problems of finding the MaxMin and MinMax area triangulations of general point sets in the plane. These problems are of unknown complexity [2]. The two named triangulations are collectively called optimal area triangulations.

Examples of n-point sets ($n \geq 4$) that require arbitrarily small angles in their exact MaxMin and MinMax area triangulation exist. However, triangulations with small angles are not suitable for most practical purposes. Therefore, we are looking for an approximation of these two optimal area triangulations. The approximation should be computable in (low?) polynomial time. To build the approximation we will use two known triangulations: the Delaunay triangulation and the optimal $30°$-triangulation (as discussed in the subsequent sections) if it exists. Thus, we intend to introduce angular restrictions to the triangulations and study how these angular restrictions influence the quality (MinMax and MaxMin Area) of the triangulation.

Throughout this text we will use β to denote the smallest angle in the optimal area triangulation (either MinMax or MaxMin). We do not know what the value of β is since we cannot solve the problem exactly. We want to approximate the optimal triangulation, which is a β-triangulation (triangulation of which all angles are greater than or equal to β), by another triangulation, which is “fatter”, i.e. has larger value of the minimum angle. We will denote this value by α. Thus the approximation will be an α-triangulation, where $\alpha > \beta$. Naturally, $\alpha \in (0°, 60°]$.

Here we will just note that if α^* is the smallest angle of the Delaunay triangulation, then α-triangulations exist if and only if $\alpha \leq \alpha^*$. The Delaunay triangulation can be computed in $O(n \log n)$ time and $O(n)$ space. Further, any relaxation of the Delaunay triangulation based on edge flips can be computed in $O(n^2)$ time and linear space. In the case when $\alpha^* \geq 30°$, the exact optimal area triangulation(s) can be computed in $O(n^3)$ time and $O(n^2)$ space by a modified Klincsek algorithm [4], based on the fact that the relative neighborhood graph is part of these triangulations [3].

2. Angular restrictions and forbidden zones. Given the fact that all angles of the α-triangulation will be larger than α, we can define a region, called forbidden zone, surrounding each possible edge between a pair of points of the given point set. The forbidden zone of an edge is by definition a region that is empty of points of the original point set if the edge is in any α-triangulation. This is a polygonal region, recursively defined by adding to the edge isosceles triangles with the edge itself is a base, and base angles of α, and continuing this process outwards of the already tiled area. The first four steps are shown in Figure 1. The parameters of the forbidden zone are fully determined by the length of the edge a and the angle α. The forbidden zone entirely contains a trapezoid with the given edge as a base, base angles of 3α and heighth of $(a/2)\tan\alpha$. The zone also entirely contains a circle surrounding each of the endpoints of the edge.

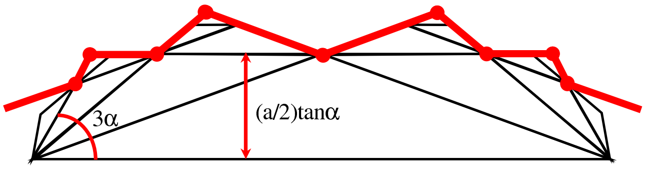

Fig. 1. Recursive construction, boundary and parameters of the forbidden zone

3. Matching triangles, cases. To evaluate the approximation ratio between a known α-triangulation and the optimal β-triangulation, we shall to use a result by Aichholzer et al. [1] that establishes the existence of a one-to-one matching between the triangles of any two triangulations of a point set. The matched triangles share at least one vertex and interior points. Based on this we have a number of possible cases that will be considered below. We shall also use the angular constraints we introduced. In an α-triangulation, all the angles are in the interval $[\alpha, 180° - 2\alpha]$. Based on the angular constraints we can identify the "forbidden zone" around each edge of the triangulation — the region of the plane that is empty of points from the original point set. For an edge of length a, the forbidden zone properly includes a trapezoid of height $\frac{a}{2} \cdot \tan\alpha$ that has base angles of 3α. The non-parallel sides of this trapezoid are, therefore, of length $\frac{a}{2} \cdot \frac{\tan\alpha}{\sin 3\alpha}$. With respect to an edge (of length a) of an α-triangulation, any other point from the set can be either outside of the strip of height $\frac{a}{2} \cdot \tan\alpha$ (Zone 1), inside a circle with radius $\frac{a}{2} \cdot \frac{\tan\alpha}{\sin 3\alpha}$ centered at one of its endpoints but outside the trapezoid (Zone 2), or inside the strip and outside the circles (Zone 3). The situation is illustrated in Figure 2.

We will denote a triangle of the α-triangulation by $\triangle ABC$ and will use the standard notation for its side lengths a, b, c. Similarly the matching triangle of

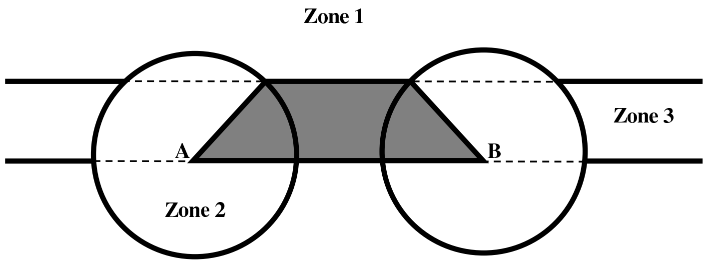

Fig. 2. The forbidden zone of the edge AB and zones 1, 2, and 3

the β-triangulation will be $\triangle A_1B_1C_1$ and the sides will be a_1, b_1, c_1. We use the following two formulae for the area of a triangle: two sides and the angle between them $A_\triangle = \frac{ab}{2}\sin\theta$ or a side and three angles $A_\triangle = \frac{a^2}{2}\cdot\frac{\sin\phi\sin\psi}{\sin\theta}$ (here ϕ, ψ, θ are the angles opposite sides a, b, c, of the triangle, respectively) which is equivalent to $A_\triangle = \frac{a^2}{2}\cdot\frac{\sin\phi\sin\psi}{\sin(\phi+\psi)}$.

Lemma 1. *Given a triangle with a side a, all the angles of which are greater than or equal to α, the minimal and maximal area of such a triangle are given by: $A_{\min} = \frac{a^2}{4}\cdot\tan\alpha$ (occurring when the triangle is an isosceles with both base angles equal to α) and $A_{\max} = \frac{a^2}{4}\cdot\cot\frac{\alpha}{2} = \frac{a^2}{4}\cdot\frac{1}{\tan\frac{\alpha}{2}}$ (occurring when the triangle is an isosceles with a top angle of α).*

We now consider the cases as to how the matched triangles from the α-triangulation and β-triangulation interact.

3.1. Three shared vertices. Trivial: the triangles are the same, the ratio of areas is equal to one.

3.2. Two shared vertices (shared edge) Assume that the two matched triangles share an edge of length a. According to Lemma 1:

$$(A_\alpha)_{\min} = \frac{a^2}{4}\cdot\tan\alpha, \qquad (A_\alpha)_{\max} = \frac{a^2}{4}\cdot\frac{1}{\tan\frac{\alpha}{2}},$$

$$(A_\beta)_{\min} = \frac{a^2}{4}\cdot\tan\beta, \qquad (A_\beta)_{\max} = \frac{a^2}{4}\cdot\frac{1}{\tan\frac{\beta}{2}}$$

Therefore, we can compute:

$$\left(\frac{A_\alpha}{A_\beta}\right)_{\min} \geq \frac{(A_\alpha)_{\min}}{(A_\beta)_{\max}} \geq \tan\alpha \cdot \tan\frac{\beta}{2}, \quad \left(\frac{A_\alpha}{A_\beta}\right)_{\max} \leq \frac{(A_\alpha)_{\max}}{(A_\beta)_{\min}} \leq \frac{1}{\tan\beta \cdot \tan\frac{\alpha}{2}}$$

3.3. Exactly one shared vertex. Assume that the two matched triangles share the vertex A (or $A \equiv A_1$). Then depending on the mutual position of the vertices B, C, B_1, C_1 we can have two different situations, as illustrated in Figure 3. Namely, exactly one side intersecting two of the sides of the matched triangle (as shown on the left) or two pairs of mutually intersecting sides (as shown on the right).

Crucial for all sub-cases that arise in this case is the intersection between a pair of sides, one from each of the matched triangles.

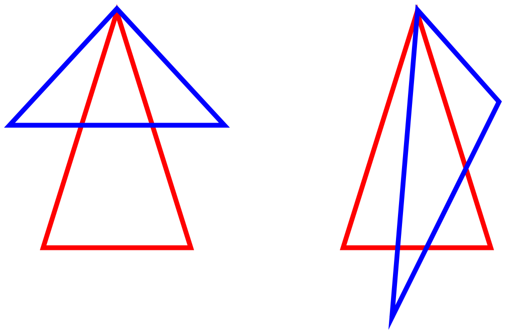

Fig. 3. Exactly one shared vertex

3.3.1. A pair of intersecting sides. Let the sides $AC = b$ and $B_1C_1 = a_1$ intersect at the point X, and in addition both pairs of vertices are outside the strip of the other edge (Zone 1). Please refer to Figure 4 for an illustration. Then, using the fact that the points C and A are outside of the forbidden zone of the edge B_1C_1, and the fact that the forbidden zone has a width of $\frac{a_1}{2} \cdot \tan\beta$, we have: $XC > \dfrac{a_1}{2} \cdot \tan\beta$ and $XA > \dfrac{a_1}{2} \cdot \tan\beta$, but $XC + XA = CA = b$, therefore: $b > a_1 \cdot \tan\beta$. We can rewrite this as $a_1 < \dfrac{b}{\tan\beta}$. Similarly, because of the fact that points C_1 and B_1 are outside of the forbidden zone of the edge AC, and the fact that the forbidden zone has a width of $\dfrac{b}{2} \cdot \tan\alpha$, we have: $XB_1 > \dfrac{b}{2} \cdot \tan\alpha$ and $XC_1 > \dfrac{b}{2} \cdot \tan\alpha$, but $XB_1 + XC_1 = B_1C_1 = a_1$, therefore: $a_1 > b \cdot \tan\alpha$. We can rewrite this as $b < \dfrac{a_1}{\tan\alpha}$. Using these we can obtain the following bounds for

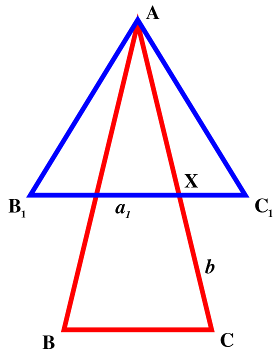

Fig. 4. A pair of intersecting sides B_1C_1 and AC

the area of the β-triangle:

$$(A_\beta)_{\min} = \frac{a_1^2}{4} \cdot \tan\beta \geq \frac{b^2}{4} \cdot \tan^2\alpha \cdot \tan\beta,$$

$$(A_\beta)_{\max} = \frac{a_1^2}{4} \cdot \frac{1}{\tan\frac{\beta}{2}} \leq \frac{b^2}{4} \cdot \frac{1}{\tan^2\beta \cdot \tan\frac{\beta}{2}}$$

Then using the results of Lemma 1 for $(A_\alpha)_{\min}$ and $(A_\alpha)_{\max}$ we can compute the bounds for the ratio of the areas as follows:

$$\left(\frac{A_\alpha}{A_\beta}\right)_{\min} \geq \frac{(A_\alpha)_{\min}}{(A_\beta)_{\max}} \geq \tan\alpha \cdot \tan^2\beta \cdot \tan\frac{\beta}{2},$$

$$\left(\frac{A_\alpha}{A_\beta}\right)_{\max} \leq \frac{(A_\alpha)_{\max}}{(A_\beta)_{\min}} \leq \frac{1}{\tan\beta \cdot \tan^2\alpha \cdot \tan\frac{\alpha}{2}}$$

Once again, these are the best bounds that we will be able to obtain when only one of the sides of the β-triangle intersects a side (or more exactly two of the sides) of the α-triangle, as shown on the left in Figure 3 and in Figure 4.

3.3.2. Two pairs of intersecting sides. As Figure 5 and the right part of Figure 3 show, we can have three intersections between the sides of the matched triangles. Then we will obtain better bounds, based on the formula for the area that uses the product of two sides and the included angle. As shown in Figure 5, the sides $BC = a$ and $A_1B_1 = c_1$ intersect. Again, assuming that both pairs of vertices are in Zone 1 with respect to the edge formed by the other two, we will have: $b \cdot \tan\alpha < a_1 < \dfrac{b}{\tan\beta}$ and $a \cdot \tan\alpha < c_1 < \dfrac{a}{\tan\beta}$.

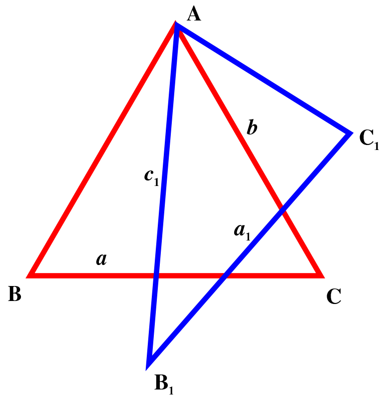

Fig. 5. Two pairs of intersecting sides B_1C_1, AC and A_1B_1, BC

Also, for an α-triangle with sides a, b and angles restricted to be larger than α, the minimum and maximum area are given by: $\frac{ab}{2} \cdot \sin\alpha \leq A_\alpha \leq \frac{ab}{2}$. That is, the minimum area is $\frac{ab}{2} \cdot \sin\alpha$, when the angle at point C is exactly α, and when the angle at C is right, we have the maximum area of $\frac{ab}{2}$. Similarly, for the β-triangle $\triangle A_1B_1C_1$: $\frac{a_1c_1}{2} \cdot \sin\beta \leq \frac{a_1c_1}{2}$. Substituting, we obtain:

$$(A_\beta)_{\min} \geq \frac{ab}{2} \cdot \tan^2\alpha \cdot \sin\beta, \quad (A_\beta)_{\max} \leq \frac{ab}{2} \cdot \frac{1}{\tan^2\beta}$$

Thus, the bounds in this case are:

$$\left(\frac{A_\alpha}{A_\beta}\right)_{\min} \geq \frac{(A_\alpha)_{\min}}{(A_\beta)_{\max}} \geq \sin\alpha \cdot \tan^2\beta, \quad \left(\frac{A_\alpha}{A_\beta}\right)_{\max} \leq \frac{(A_\alpha)_{\max}}{(A_\beta)_{\min}} \leq \frac{1}{\sin\beta \cdot \tan^2\alpha}$$

Recall that we are assuming $\beta \leq 30°$. To show that these bounds are better than the ones derived with only one pair of intersecting sides (in Section 3.3.1), consider the inequalities:

$$\tan\alpha \cdot \tan^2\beta \cdot \tan\frac{\beta}{2} < \sin\alpha \cdot \tan^2\beta \Leftrightarrow \tan\frac{\beta}{2} < \cos\alpha,$$

which is true whenever $\alpha < \arccos(\tan 15°) \simeq 74.45°$, since $\tan\frac{\beta}{2} < \tan 15° <$

$\cos\alpha$, for the lower bound, and

$$\frac{1}{\sin\beta\cdot\tan^2\alpha} < \frac{1}{\tan\beta\cdot\tan^2\alpha\cdot\tan\frac{\alpha}{2}}$$
$$\Leftrightarrow \tan\beta\cdot\tan^2\alpha\cdot\tan\frac{\alpha}{2} < \sin\beta\cdot\tan^2\alpha \Leftrightarrow \tan\frac{\alpha}{2} < \cos\beta,$$

which is true whenever $\beta < \arccos(\tan 30°) \simeq 54.74°$, since $\tan\frac{\alpha}{2} < \tan 30° < \cos\beta$, for the upper bound. This concludes the cases when all the vertices of the two triangles are situated so that they lie in Zone 1 with respect to the edges of the other triangle. From this point on, at least one of the points will be in Zone 2 or Zone 3 with respect to some edge of the other triangle. When we have exactly one of the points in Zone 2, we can show that the area of the triangle will be dominated from both above and below by other possible triangles that belong to the cases considered in the following section. The same is true if at least one point is in Zone 3. Therefore, the only cases remaining are those when exactly two points lie in Zone 2.

3.3.3. Placements of points in Zone 2. When vertex A is shared, the two points B_1 and C_1 can be placed in the circles defining Zone 2, centered at B and C, in four different ways. Here we recall that the radius of the circle defining Zone 2 for an edge of length a is $r = \frac{a}{2}\cdot\frac{\tan\alpha}{\sin 3\alpha}$.

For convenience, we will introduce the constant $k = \frac{1}{2}\cdot\frac{\tan\alpha}{\sin 3\alpha}$, thus $r = k\cdot a$. The four possible distributions of the points in the circles are illustrated in Figure 6. We will consider the possible placements and derive bounds for the ratio of the areas. Starting with the placement **(a)**, where we can use the approach of the Section 3.3.2 by constraining two of the sides of the β-triangle, namely $b - r \le b_1 \le b + r$, and having in mind that $r = k\cdot b$ we obtain $b(1-k) \le b_1 \le b(1+k)$, and similarly $c(1-k) \le c_1 \le c(1+k)$. For the areas we have: $\frac{bc}{2}\cdot\sin\alpha \le A_\alpha \le \frac{bc}{2}$, $\frac{b_1c_1}{2}\cdot\sin\beta \le A_\beta \le \frac{b_1c_1}{2}$, and therefore

$$(A_\alpha)_{\min} = \frac{bc}{2}\cdot\sin\alpha, \qquad (A_\alpha)_{\max} = \frac{bc}{2},$$

$$(A_\beta)_{\min} \ge \frac{bc}{2}(1-k)^2\cdot\sin\beta, \quad (A_\beta)_{\max} \le \frac{bc}{2}(1+k)^2.$$

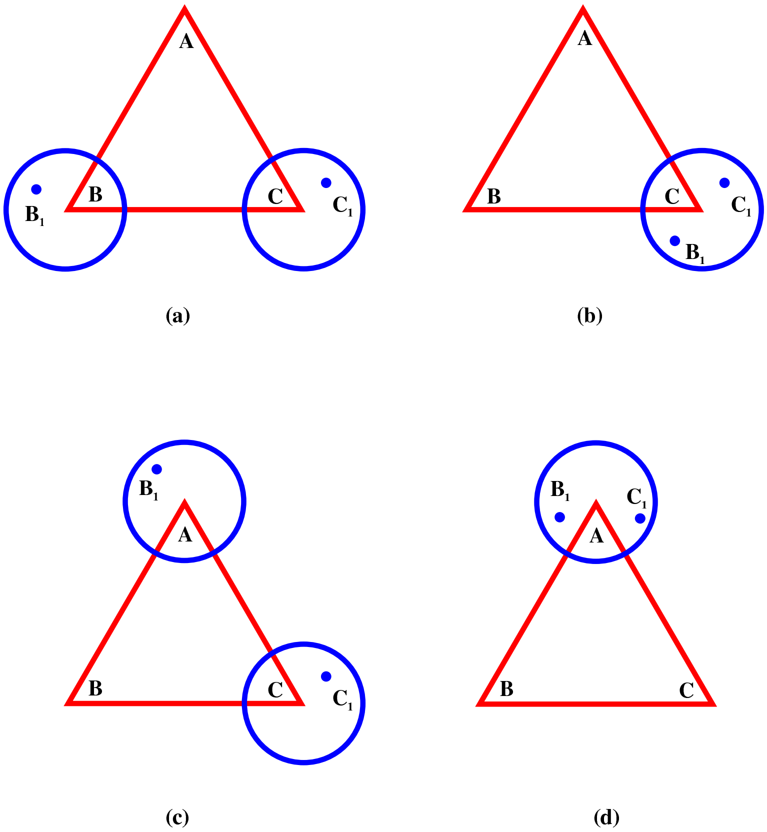

Fig. 6. Possible placements of B_1 and C_1 in zone 2

Thus, for the ratio of the areas we have:

$$\left(\frac{A_\alpha}{A_\beta}\right)_{\min} \geq \frac{(A_\alpha)_{\min}}{(A_\beta)_{\max}} \geq \frac{\frac{bc}{2} \cdot \sin\alpha}{\frac{bc}{2}(1+k)^2} \geq \frac{\sin\alpha}{(1+k)^2},$$

$$\left(\frac{A_\alpha}{A_\beta}\right)_{\max} \leq \frac{(A_\alpha)_{\max}}{(A_\beta)_{\min}} \leq \frac{\frac{bc}{2}}{\frac{bc}{2}(1-k)^2 \cdot \sin\beta} \leq \frac{1}{(1-k)^2 \cdot \sin\beta}$$

In placement **(b)**, we can constrain two of the sides of the β-triangle with relation to only one side of the α-triangle: $b - r \leq b_1 \leq b + r$, $b - r \leq c_1 \leq b + r$, which leads to the following: $b(1-k) \leq b_1 \leq b(1+k)$, $b(1-k) \leq c_1 \leq b(1+k)$. Further

$\frac{b^2}{4} \cdot \tan\alpha \leq A_\alpha \leq \frac{b^2}{4} \cdot \frac{1}{\tan\frac{\alpha}{2}}$, $\frac{b_1 c_1}{2} \cdot \sin\beta \leq A_\beta \leq \frac{b_1 c_1}{2}$, and therefore

$$(A_\beta)_{\min} \geq \frac{b^2}{2}(1-k)^2 \cdot \sin\beta, \qquad (A_\beta)_{\max} \leq \frac{b^2}{2}(1+k)^2.$$

For the bounds we obtain:

$$\left(\frac{A_\alpha}{A_\beta}\right)_{\min} \geq \frac{(A_\alpha)_{\min}}{(A_\beta)_{\max}} \geq \frac{\frac{b^2}{4} \cdot \tan\alpha}{\frac{b^2}{2}(1+k)^2} \geq \frac{\tan\alpha}{2(1+k)^2},$$

$$\left(\frac{A_\alpha}{A_\beta}\right)_{\max} \leq \frac{(A_\alpha)_{\max}}{(A_\beta)_{\min}} \leq \frac{\frac{b^2}{4} \cdot \frac{1}{\tan\frac{\alpha}{2}}}{\frac{b^2}{2}(1-k)^2 \cdot \sin\beta} \leq \frac{1}{2(1-k)^2 \cdot \sin\beta \cdot \tan\frac{\alpha}{2}}$$

Comparing these bounds with the bounds for placement **(a)**, we can see that:

$$\frac{\tan\alpha}{2(1+k)^2} < \frac{\sin\alpha}{(1+k)^2} \Leftrightarrow \frac{1}{2} < \cos\alpha,$$

which is true whenever $\alpha < 60^\circ$, for the lower bound, and

$$\frac{1}{(1-k)^2 \cdot \sin\beta} < \frac{1}{2(1-k)^2 \cdot \sin\beta \cdot \tan\frac{\alpha}{2}} \Leftrightarrow \tan\frac{\alpha}{2} < \frac{1}{2},$$

which is true whenever $\alpha < 2\arctan\left(\frac{1}{2}\right) \simeq 53.13^\circ$, for the upper bound. Thus, the bounds in placement **(b)** are always worse than those in placement **(a)**.

In the third placement, **(c)** we again are going to relate two of the sides of the β-triangle to only one side of the α-triangle: $b-2r \leq a_1 \leq b+2r$, $b-r \leq b_1 \leq b+r$, or equivalently: $b(1-2k) \leq a_1 \leq b(1+2k)$, $b(1-k) \leq b_1 \leq b(1+k)$. Similar to the analysis of the previous placement, for the areas we have: $\frac{b^2}{4} \cdot \tan\alpha \leq A_\alpha \leq \frac{b^2}{4} \cdot \frac{1}{\tan\frac{\alpha}{2}}$, $\frac{a_1 b_1}{2} \cdot \sin\beta \leq A_\beta \leq \frac{a_1 b_1}{2}$, and therefore

$$(A_\beta)_{\min} \geq \frac{b^2}{2}(1-k)(1-2k) \cdot \sin\beta, \qquad (A_\beta)_{\max} \leq \frac{b^2}{2}(1+k)(1+2k).$$

The bounds are:

$$\left(\frac{A_\alpha}{A_\beta}\right)_{\min} \geq \frac{(A_\alpha)_{\min}}{(A_\beta)_{\max}} \geq \frac{\tan\alpha}{2(1+k)(1+2k)},$$

$$\left(\frac{A_\alpha}{A_\beta}\right)_{\max} \leq \frac{(A_\alpha)_{\max}}{(A_\beta)_{\min}} \leq \frac{1}{2(1-k)(1-2k) \cdot \sin\beta \cdot \tan\frac{\alpha}{2}}$$

Again, we are going to compare these to the bounds obtained for placement **(b)**. We have:

$$\frac{\tan\alpha}{2(1+k)(1+2k)} < \frac{\tan\alpha}{2(1+k)^2} \Leftrightarrow 1+k < 1+2k,$$

which is true whenever $k > 0$, for the lower bound, and

$$\frac{1}{2(1-k)^2 \cdot \sin\beta \cdot \tan\frac{\alpha}{2}} < \frac{1}{2(1-k)(1-2k) \cdot \sin\beta \cdot \tan\frac{\alpha}{2}} \Leftrightarrow 1-2k < 1-k,$$

which is true whenever $k > 0$ and

$$1-2k > 0 \Leftrightarrow k < \frac{1}{2} \Leftrightarrow \frac{\tan\alpha}{2\sin 3\alpha} < \frac{1}{2} \Leftrightarrow \tan\alpha < \sin 3\alpha$$

The last inequality can be shown to be valid in the interval $\alpha \in (0°, 30°]$, which is sufficient for our considerations. Thus, the bounds in placement **(c)** are always worse than those in placement **(b)**. Note that for the three placements considered so far, the lower bound (the one that gives us the approximation constant for the MaxMin Area triangulation) does not depend on the angle β.

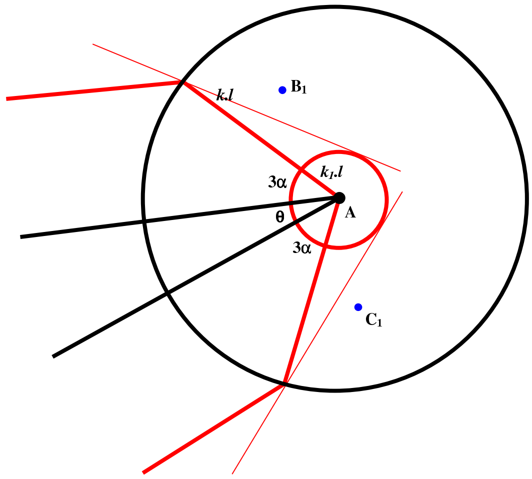

Fig. 7. Two constraining circles in placement (d)

Finally, the placement **(d)** should be considered. There, the points B_1 and C_1 are in the circle defining zone 2 for the shared vertex A. This is the worst case, as B_1 and C_1 can be "very close" to A, thus making the area of the β-triangle small, and increasing the upper bound (the one related to the approximation constant for the MinMax Area triangulation). However, if we include the forbidden zones of the edges in the vicinity of the point A, we can derive some constraints. Without

loss of generality we can assume that $b \geq c$ for the two edges incident to A, $AB = c$ and $AC = b$. First, the angle of the α-triangle at A, $\angle BAC \geq \alpha$ by our assumption. Moreover, the trapezoid parts of the forbiden zones of the edges BA and CA form angles of 3α with these edges at A. Therefore, there is a circular sector with a central angle of 7α that is forbidden for the points B_1 and C_1. Hence, the top angle of the β-triangle is $\angle B_1AC_1 \geq 7\alpha$. The first constraint therefore is $2\beta + 7\alpha \leq 180°$, using the sum of the angles of the triangle $\triangle B_1AC_1$, which is a β-triangle. Recall that $\beta < \alpha$, thus this is only possible for $\beta < 20°$.
To constrain the lengths of the sides B_1A and C_1A of the β-triangle, we are going to use the fact that the forbidden zones of edges AB and AC contain a circle around A. If $i = \left\lceil \frac{180°}{2\alpha} - \frac{1}{2} \right\rceil$ it can be shown that the radius of the forbidden circle is $k_1 b$ for the edge AC (respectively $k_1 c$ for the edge AB), where $k_1 = \frac{d_i}{l} = \frac{1}{(2\cos\alpha)^i}$. The lengths of the sides B_1A and C_1A of the β-triangle are therefore constrained to $k_1 c \leq b_1 \leq kb$, $k_1 c \leq c_1 \leq kb$. Therefore, for the areas of the two triangles we have: $\frac{b^2}{4} \cdot \tan\alpha \leq A_\alpha \leq \frac{c^2}{4} \cdot \frac{1}{\tan\frac{\alpha}{2}}$, $\frac{b_1 c_1}{2} \cdot \sin 2\beta \leq A_\beta \leq \frac{b_1 c_1}{2}$, and therefore

$$(A_\beta)_{\min} \geq \frac{c^2}{2} k_1^2 \cdot \sin 2\beta, \qquad (A_\beta)_{\max} \leq \frac{b^2}{2} k^2.$$

The bounds in this case are:

$$\left(\frac{A_\alpha}{A_\beta}\right)_{\min} \geq \frac{(A_\alpha)_{\min}}{(A_\beta)_{\max}} \geq \frac{\frac{b^2}{4} \cdot \tan\alpha}{\frac{b^2}{2} k^2} \geq \frac{\tan\alpha}{2k^2},$$

$$\left(\frac{A_\alpha}{A_\beta}\right)_{\max} \leq \frac{(A_\alpha)_{\max}}{(A_\beta)_{\min}} \leq \frac{\frac{c^2}{4} \cdot \frac{1}{\tan\frac{\alpha}{2}}}{\frac{c^2}{2} k_1^2 \cdot \sin 2\beta} \leq \frac{1}{2k_1^2 \cdot \sin^2\beta \cdot \tan\frac{\alpha}{2}}$$

It can be seen that when, depending on α and β, this case is possible, the bounds are going to be worse than those of the other cases.

4. Algorithmic results and sample values. In Section 3 we considered possible cases of matched triangles and positions of the points with respect to the forbidden zones. Based on this we derived the following bounds of the approximation factors for the MaxMin area triangulation:

$$f_1 = \max\left(\frac{1}{\tan\alpha \tan^2\beta \tan\frac{\beta}{2}}, \frac{2(1+k)(1+2k)}{\tan\alpha}, \frac{2k^2}{\tan\alpha}\right)$$

and for the MinMax area triangulation:

$$f_2 = \max\left(\frac{1}{\tan\beta\tan^2\alpha\tan\frac{\alpha}{2}}, \frac{1}{2(1-k)(1-2k)\sin\beta\tan\frac{\alpha}{2}}, \frac{1}{2k_1^2\cdot\sin^2\beta\cdot\tan\frac{\alpha}{2}}\right)$$

The approximation factor f_1 shows how many times the smallest area triangle in the approximating α-triangulation is smaller than the smallest area triangle in the optimal (MaxMin area) triangulation. Similarly, f_2 gives the ratio of the largest area triangle in the approximating triangulation, compared to the largest area triangle in the optimal (MinMax area) triangulation.

As mentioned earlier, we can compute the optimal 30°-triangulation (if it exists) by modified Klincsek's algorithm [4] in $O(n^3)$ time and $O(n^2)$ space, using the fact that the Relative Neighbourhood Graph is a part of it [3]. Alternatively, we can relax Delaunay by area equalizing flips, which will take $O(n^2)$ time and $O(n)$ space. Thus we achieve a (sub)cubic time algorithm that approximates the optimal area triangulations, by the above given factors. The value of α can be chosen from practical considerations. Sample results are presented in Table 1.

Table 1. Sample values for f_1 and f_2

α	30	30	25	25	20	20	15
β	25	20	20	15	15	10	10
f_1	35.930	74.149	91.807	226.87	290.66	1010.1	1372.0
f_2	24.010	30.716	56.994	77.418	311.28	455.06	7900.1

Values obtained for the approximation factors show two interesting trends. The approximation factor for the MaxMin area triangulation, f_1, is much more dependent on the difference between α and β. The approximation factor for the MinMax area triangulation, f_2, initially increases slower than f_1 with the decrease of the angle α, however once a threshold is reached, i.e. once the placement **(d)** becomes posible, the value of f_2 jumps up sharply. There seems to be no threshold value of α or specifically bad placement of points for the MaxMin area triangulation. This partly supports the conjecture of Edelsbrunner [2] that the MinMax area triangulation of a point set is computationally harder problem than the MaxMin area triangulation in general.

Note that in practice the approximation factor will be smaller, due to the fact that our case analysis does not take into account the "real" matching, i.e., the ratio between the areas of the respective worst (either smallest or largest area) triangles. Thus, our results should be treated as upper bounds on the approximation factors. We expect that improvement in these bounds is possible.

One way to improve the bounds is to consider specific types of matchings in which triangles are ordered by area and then matched against the triangles of the other triangulation.

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Received February 22, 2010
Final Accepted July 23, 2010