

A NEW APPROACH TO FUZZY ARITHMETIC*

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ABSTRACT. This work shows an application of a generalized approach for constructing dilation-erosion adjunctions on fuzzy sets. More precisely, operations on fuzzy quantities and fuzzy numbers are considered. By the generalized approach an analogy with the well known interval computations could be drawn and thus we can define outer and inner operations on fuzzy objects. These operations are found to be useful in the control of bioprocesses, ecology and other domains where data uncertainties exist.

1. Introduction. There are several approaches for fuzzifying mathematical morphology, see for instance [1, 4]. In our work we step on the framework of Deng and Heijmans (see for details [3]) based on adjoint fuzzy logical operators – conjunctors and implicators. We generalize this definition presenting a universal framework. Thus we can define naturally fuzzy geodesic morphological operations [10]. Also, this model is applicable to fuzzy arithmetic, built by

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analogy with the interval arithmetic [9] which makes possible the definition of inner addition and multiplication of fuzzy numbers. Inner and outer operations on other kind of fuzzy quantities like fuzzy vectors (dot and cross product), fuzzy complex numbers etc. could be considered as well.

In this work we use the same notions and notations about complete lattices and the morphological operations on them as in [5]. For instance, let \mathcal{L} be a complete lattice with a supremum generating family l , and let T be an Abelian group of automorphisms of \mathcal{L} acting transitively over l . The elements of T are denoted by τ_x , namely for any $x \in l$, $\tau_x(o) = x$, where o is a fixed element of l interpreted as an origin. Then also, we can consider a symmetry in \mathcal{L} as $\check{A} = \bigvee_{a \in l(A)} \tau_a^{-1}(o)$. Evidently $\check{a} = \tau_a^{-1}(o) = \tau_{\check{a}}(o)$ for any $a \in l$. If A is an arbitrary element of the lattice \mathcal{L} let us denote by $l(A) = \{a \in l \mid a \leq A\}$ the supremum-generating set of A . Following [5] we define the operations

$$(1) \quad \delta_A = \bigvee_{a \in l(A)} \tau_a$$

and

$$(2) \quad \varepsilon_A = \bigwedge_{a \in l(A)} \tau_a^{-1} = \bigwedge_{a \in l(A)} \tau_{\check{a}}$$

which form an adjunction. δ_A and ε_A are T -invariant operators called *dilation and erosion by the structuring element A*. Recall that a pair of operators (ε, δ) between two lattices, $\varepsilon : \mathcal{M} \mapsto \mathcal{L}$ and $\delta : \mathcal{L} \mapsto \mathcal{M}$, is called an *adjunction* if for every two elements $X \in \mathcal{L}$ and $Y \in \mathcal{M}$ it follows that

$$\delta(X) \leq Y \iff X \leq \varepsilon(Y).$$

In [5] it is proved that if (ε, δ) is an adjunction then ε is erosion and δ is dilation. On the other hand, every dilation $\delta : \mathcal{L} \mapsto \mathcal{M}$ has a unique adjoint erosion $\varepsilon : \mathcal{M} \mapsto \mathcal{L}$, and vice-versa.

In the case of standard morphological operations when the lattice \mathcal{L} is made of the subsets of the Euclidean space \mathbf{R}^d we shall use the standard notations for Minkowski subtraction and addition for the adjoint pair $(\varepsilon_A, \delta_A)$ [5, 11]:

$$\delta_A(X) = X \oplus A = \{x + a \mid x \in X, a \in A, \}$$

$$\varepsilon_A(X) = X \ominus A = \{y \mid A + y \subseteq X\}.$$

2. Interval operations and mathematical morphology. Following [7] we introduce the following notations: Let $\mathcal{L} = I(\mathbf{R})$ be the family of

all closed finite intervals of the real line. For completeness we may assume that the empty set \emptyset is an element of $I(\mathbf{R})$. The non-empty finite intervals from $I(\mathbf{R})$ are denoted by Roman capitals, namely $A = [a^-, a^+]$, i.e. a^- is the left endpoint of the interval A and a^+ is its right endpoint such that $a^- \leq a^+$. By $\omega(A)$ we denote the length of the interval A , $\omega(A) = a^+ - a^-$.

Consider the outer and inner interval additive operations as defined by S. Markov in [7]:

$$\begin{aligned} A + B &= [a^- + b^-, a^+ + b^+], & A +^- B &= [a^{-s} + b^s, a^s + b^{-s}], \\ A - B &= [a^- - b^+, a^+ - b^-], & A -^- B &= [a^{-s} - b^{-s}, a^s - b^s], \end{aligned}$$

where

$$s = \begin{cases} +, & \omega(A) \geq \omega(B), \\ -, & \omega(A) < \omega(B). \end{cases}$$

Here a^s with $s \in \{+, -\}$ denotes a certain endpoint of the interval A : the left one if $s = -$ and the right one if $s = +$. As proved in [9] the outer and inner interval operations in $I(\mathbf{R})$ are related to dilations and erosions as follows:

$$(3) \quad A + B = A \oplus B = \delta_A(B) = \delta_B(A),$$

$$(4) \quad A +^- B = A \ominus (-B) \cup B \ominus (-A) = \varepsilon_{-B}(A) \cup \varepsilon_{-A}(B).$$

It is easy to demonstrate that for the outer and inner subtraction we have the following relations [9]:

$$(5) \quad A - B = A + (-B) = A \oplus (-B),$$

$$(6) \quad A -^- B = A +^- (-B) = A \ominus B \cup -(B \ominus A).$$

In many applications such as locating the roots of a polynomial with interval coefficients, it is desirable to also spread the multiplicative operations over the set of real intervals. Let \mathcal{L} be the set of all closed finite and infinite real intervals different from the singleton $[0, 0]$. The order relation is the set inclusion, while the supremum is defined as the closed convex hull of the union, and the infimum is defined as intersection. There is an Abelian group of automorphisms

$$T = \{\tau_h \mid h \in \mathbf{R} \setminus \{0\}, \tau_h(\{x\}) = \{xh\}\},$$

which acts transitively on l and $\tau_h(A) = [\min(a^-h, a^+h), \max(a^-h, a^+h)]$. Then we can define dilation and erosion operations by the structuring element A using expressions (1) and (2):

$$\delta_A(B) = \bigvee_{a \in l(A)} \tau_a(B), \quad \varepsilon_A(B) = \bigwedge_{a \in l(A)} \tau_{1/a}(B).$$

Having in mind outer and inner multiplications in $I(\mathbf{R})$ as defined in [7] by Markov, we claim that [9]:

- For the dilation defined above we have $\delta_B(A) = \delta_A(B) = A \times B$.
- Denoting $\frac{1}{B} = \left[\frac{1}{b^+}, \frac{1}{b^-} \right]$, we have $A \times^- B = \varepsilon_{\frac{1}{A}}(B) \cup \varepsilon_{\frac{1}{B}}(A)$ for any intervals A and B not containing zero (denoting by \mathcal{Z} the collection of intervals containing 0) under the inner multiplication operation “ \times^- ” defined in [7], which can be represented as:

$$(7) \quad A \times^- B = \begin{cases} [a^{\sigma(B)\epsilon} b^{-\sigma(A)\epsilon}, a^{-\sigma(B)\epsilon} b^{\sigma(A)\epsilon}], & \epsilon = \psi(A, B), \quad A, B \in \mathcal{L} \setminus \mathcal{Z}, \\ [a^{-\sigma(A)} b^{-\sigma(A)}, a^{-\sigma(A)} b^{\sigma(A)}], & A \in \mathcal{L} \setminus \mathcal{Z}, \quad B \in \mathcal{Z} \\ [a^{-\sigma(B)} b^{-\sigma(B)}, a^{\sigma(B)} b^{-\sigma(B)}], & A \in \mathcal{Z}, B \in \mathcal{L} \setminus \mathcal{Z}, \\ [\max(a^- b^+, a^+ b^-), \min(a^- b^-, a^+ b^+)], & A, B \in \mathcal{Z}. \end{cases}$$

Here $\psi(A, B) = +$ if $\chi(A) \geq \chi(B)$ and $\psi(A, B) = -$ otherwise, where $\chi(A) = a^-/a^+$ if $a^- + a^+ \geq 0$ and $\chi(A) = a^+/a^-$ if $a^- + a^+ < 0$.

If A doesn't contain zero

$$\sigma(A) = \begin{cases} +, & \text{if } 0 < a^-; \\ -, & \text{if } a^+ < 0. \end{cases}$$

It is proved that $A +^- B \subseteq A + B$ and $A \times^- B \subseteq A \times B$ [7, 9].

3. Fuzzy sets and fuzzy morphological operations. Consider the set E called the universal set. A fuzzy subset A of the universal set E can be considered as a function $\mu_A : E \mapsto [0, 1]$, called the membership function of A . $\mu_A(x)$ is called the degree of membership of the point x to the set A . The ordinary subsets of E , sometimes called ‘crisp sets’, can be considered as a particular case of a fuzzy set with membership function taking only the values 0 and 1.

Let $0 < \alpha \leq 1$. An α -cut of the set X (denoted by $[X]_\alpha$) is the set of points x , for which $\mu_X(x) \geq \alpha$.

The usual set-theoretical operations can be defined naturally on fuzzy sets: Union and intersection of a collection of fuzzy sets is defined as supremum, resp. infimum of their membership functions. Also, we say that $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ for all $x \in E$. The complement of A is the set A^c with membership function $\mu_{A^c}(x) = 1 - \mu_A(x)$ for all $x \in E$. If the universal set E is linear,

like the d -dimensional Euclidean vector space \mathbf{R}^d or the space of integer vectors with length d , then any geometrical transformation arising from a point mapping can be generalised from sets to fuzzy sets by taking the formula of this transformation for graphs of numerical functions, i.e. for any transformation ψ like scaling, translation, rotation etc. we have that $\psi(\mu_A(x)) = \mu_A(\psi^{-1}(x))$. Therefore we can transform fuzzy sets by transforming their α -cuts like ordinary sets. Further on, for simplicity, we shall write simply $A(x)$ instead of $\mu_A(x)$.

Say that the function $c(x, y) : [0, 1] \times [0, 1] \mapsto [0, 1]$ is conjunctor if c is increasing in both arguments, $c(0, 1) = c(1, 0) = 0$, and $c(1, 1) = 1$. We say that a conjunctor is a t -norm if it is commutative, i.e. $c(x, y) = c(y, x)$, associative $c(c(x, y), z) = c(x, c(y, z))$ and $c(x, 1) = x$ for every number $x \in [0, 1]$, see for instance [1, 8].

Say that the function $i(x, y) : [0, 1] \times [0, 1] \mapsto [0, 1]$ is an implicator if i is increasing in y and decreasing in x , $i(0, 0) = i(1, 1) = 1$, and $i(1, 0) = 0$.

In [3] a number of conjunctor–implicator pairs are proposed. Here we give examples of two of them:

$$c(b, y) = \min(b, y),$$

$$i(b, x) = \begin{cases} x & x < b, \\ 1 & x \geq b \end{cases}.$$

$$c(b, y) = \max(0, b + y - 1),$$

$$i(b, x) = \min(1, x - b + 1).$$

The first pair is known as operations of Gödel–Brouwer, while the second pair is suggested by Lukasiewicz.

Also, a widely used conjunctor is $c(b, y) = by$, see [8]. Its adjoint implicator is

$$i(b, x) = \begin{cases} \min\left(1, \frac{x}{b}\right) & b \neq 0, \\ 1 & b = 0 \end{cases}.$$

3.1. General definition of fuzzy morphology. There are different ways to define fuzzy morphological operations. An immediate paradigm for defining fuzzy morphological operators is to lift each binary operator to a grey-scale operator by fuzzifying its primitive composing operations. However in this way we rarely obtain erosion–dilation adjunctions, which leads to non-idempotent openings and closings. Therefore we use the idea from [3], saying that having an adjoint conjunctor–implicator pair, we can define a fuzzy erosion–dilation adjunction.

So let us consider the universal set E and a class of fuzzy sets $\{A_y, | y \in E\}$. Then for any fuzzy subset X of the universal set E , let us define fuzzy dilation and erosion as follows:

$$(8) \quad \delta(X)(x) = \bigvee_{y \in E} c(A_x(y), X(y)),$$

$$(9) \quad \varepsilon(X)(x) = \bigwedge_{y \in E} i(A_y(x), X(y)).$$

Theorem. $(\varepsilon(X), \delta(X))$ is adjunction.

Proof. To prove that this pair of operations is an adjunction, let us consider the case $\delta(X) \subseteq Z$ in fuzzy sense, which means that for every $x, y \in E$ $c(A_x(y), X(y)) \leq Z(x)$. Then $X(y) \leq i(A_x(y), Z(x))$ for all $x, y \in E$. Since $\varepsilon(Z)(y) = \bigwedge_{x \in E} i(A_x(y), Z(x))$, then we consider that $X \subseteq \varepsilon(Z)$, which ends the proof. \square

3.2. How to define T -invariant fuzzy morphological operations?

Let us consider an universal set E . Also let there exist an abelian group of automorphisms T in $\mathcal{P}(E)$ such that T acts transitively on the supremum-generating family $l = \{\{e\} | e \in E\}$ as defined previously. In this case, for shortness we shall say that T acts transitively on E . Then having an arbitrary fuzzy subset B from E , we can define a family of fuzzy sets $\{A_y^B | y \in E\}$ such as $A_y^B(x) = B(\tau_y^{-1}(x))$. Recall that for any $\tau \in T$ there exists a unique $y \in E$ such that $\tau = \tau_y$, and for any fuzzy subset M we have that $(\tau(M))(x) = M(\tau^{-1}(x))$. Then having in mind equations (8) we can define a fuzzy adjunction by the structuring element B by:

$$(10) \quad \delta_B(X)(x) = \bigvee_{y \in E} c(A_x^B(y), X(y)),$$

$$(11) \quad \varepsilon_B(X)(x) = \bigwedge_{y \in E} i(A_y^B(x), X(y)).$$

We show that the upper operations are T -invariant. Following [5], it is sufficient to demonstrate that every such erosion commutes with an arbitrary automorphism τ_b for any $b \in E$. Evidently

$$\varepsilon_B(\tau_b(X))(x) = \bigwedge_{y \in E} i(B(\tau_y^{-1}(x)), X(\tau_b^{-1}(y))).$$

suppose that $\tau_b^{-1}(y) = z$, which means that $\tau_y = \tau_z \tau_b$. Then

$$\varepsilon_B(\tau_b(X))(x) = \bigwedge_{z \in E} i(B(\tau_z^{-1}(\tau_b^{-1}(x))), X(z)) = \varepsilon_B(X)(\tau_b^{-1}(x)),$$

which simply means that $\varepsilon_B(\tau_b(X)) = \tau_b(\varepsilon_B(X))$, which ends the proof of the proposition.

Now consider that in E we have a continuous operation $*$: $E \times E \mapsto E$. Then let us define $\tau_b(x) = b * x$. In the case of the Gödel-Brouwer conjunctor-implicator pair the respective dilation has the form

$$(\delta_B(X))(x) = \bigvee_{y*z=x} \min(X(y), B(z)).$$

If the operation $*$ is commutative, then $\delta_B(X) = \delta_X(B)$.

4. Fuzzy quantities, fuzzy numbers and operations on them.

A fuzzy set from the universal set $E = \mathbf{R}^d$ is said to be a *fuzzy quantity* if its support is bounded. A fuzzy quantity is said to be *convex* if its α -cuts are convex, which is the same as $A(y) \geq \min(A(x), A(z))$ for all y from the closed segment between x and z [8]. A convex fuzzy quantity P is a fuzzy point, or a fuzzy vector in E when its membership function is upper semicontinuous and there exists a unique point $z \in E$ such that $P(z) = 1$. Then we say that P is a fuzzy point around z . In the case $d = 1$ we talk about *fuzzy numbers* whose α -cuts are closed intervals.

Let A and B be two fuzzy numbers with α -cuts $A_\alpha = [a(\alpha)^-, a(\alpha)^+]$ and $B_\alpha = [b(\alpha)^-, b(\alpha)^+]$ for any α between zero and one. Then we can define a strong ordering relation $A \leq_s B$ such as $a(\alpha)^- \leq b(\alpha)^-$ and $a(\alpha)^+ \leq b(\alpha)^+$ for every $0 < \alpha \leq 1$, and a weak ordering $A \leq_w B$ such as $a(\alpha)^+ \leq b(\alpha)^+$ for every $0 < \alpha \leq 1$.

If given two fuzzy points P' and P'' , one can define a distance between them as follows [2]: Given a metric D in \mathbf{R}^d , let us consider for every $\alpha \in (0, 1]$ the set

$$\Omega(\alpha) = \{D(u, v) \mid u \in P'_\alpha \text{ and } v \in P''_\alpha\}.$$

Then let us define a fuzzy set by its cuts: $D^f(P', P'')_\alpha = \Omega(\alpha)$. As it is proved in [2], D^f is a nonnegative fuzzy number which nearly satisfies the condition of a distance. It is clear that $D^f(P', P'') = D^f(P'', P')$. However the triangle inequality is satisfied as a weak ordering, and $D^f(P', P'')$ is a fuzzy number around zero if and only if P' and P'' are fuzzy points around one and the same crisp point. So, it is clear that we need to exploit arithmetic operations between fuzzy numbers, and fuzzy quantities in general. For a detailed study of arithmetic operations on fuzzy numbers one can refer to [6]. In general, fuzzy arithmetic is strongly connected to decision making.

Following the *extension principle* (see [8]) for the definition of any operation $X * B$ between fuzzy sets if $*$ is a continuous binary operation in the universal set E , it has been demonstrated that $X * B = \bigvee_{y*z=x} \min(X(y), B(z))$, and therefore

$$(12) \quad [X * B]_\alpha = [X]_\alpha * [B]_\alpha = \{z \in E \mid z = a * b, a \in [X]_\alpha, b \in [B]_\alpha\}.$$

It is evident that this is the expression for the dilation in the case of Gödel-Brouwer logical operations.

Now consider the group of automorphisms $\tau_b(x) = b * x$ in \mathbf{R} and the fuzzy operations on $F(\mathbf{R})$ defined by Gödel-Brouwer's conjunctive-implicator pair:

$$(\delta_B(A))(x) = \bigvee_{y*z=x} \min(A(y), B(z)),$$

$$(\varepsilon_B(A))(x) = \inf_{y \in \mathbf{R}} (h(A(y) - B(\tau_x^{-1}(y))) (1 - A(y)) + A(y)),$$

where $h(x) = 1$ when $x \geq 0$ and is zero otherwise. These are the operations defined by (10) in this particular case.

So it is clear that if $\tau_b(x) = x + b$ and $* = +$ then

$$(\delta_B(A)) = A + B.$$

Then by analogy to the interval operations we can use (3) as definition for *inner addition* operation by $A +^- B = \varepsilon_{-B}(A) \cup \varepsilon_{-A}(B)$. If $\tau_b(x) = xb$ for $b \neq 0$ and $y * z = yz$ then $(\delta_B(A)) = A \times B$. In this case an inner multiplication exists as well:

$$A \times^- B = \varepsilon_{\frac{1}{B}}(A) \cup \varepsilon_{\frac{1}{A}}(B).$$

Note that in this definition we can work with fuzzy numbers which do not contain 0 in their support, but this is the main practically useful case in the control systems when we measure imprecisely some physical quantities. It is not difficult to show directly that $A +^- B \subseteq A + B$ and $A \times^- B \subseteq A \times B$ in a fuzzy sense[10].

5. Conclusions. We have shown that fuzzy inner and outer arithmetical operations can be represented as morphological ones, and choosing an arbitrary t -norm we can obtain a large variety of different commutative outer and inner additions and multiplications (see also [10].) Also, we can define outer and inner versions of operations on fuzzy vectors like vector (cross) product and

scalar (dot) product for solving geometric problems with uncertainties, for instance finding whether a polygon is nearly convex. An immediate result is also, that since dilation is continuous but erosion is not (it is upper semicontinuous) [5] it is seen that we can prove continuity immediately only for the outer operations both in interval and fuzzy cases, not for the inner ones. Also, for inner operations expression (12) is not valid in general. However inner operations are useful for obtaining plausible and more exact numerical solutions, therefore they are worth studying in detail [7]. Our further plans include a more detailed study of the algebraic and analytical properties of fuzzy operations such as conditions for validity of the distributive law and existence of opposite element. Also, a fuzzy control scheme based on fuzzy arithmetic for biological water cleaning shall be designed.

REFERENCES

- [1] BLOCH I., H. MAÎTRE. Fuzzy mathematical morphologies: a comparative study. *Pattern Recognition*, **28** (1995), No 9, 1341–1387.
- [2] BUCKLEY J. J., E. ESLAMI. Fuzzy plane geometry I: Points and lines. *Fuzzy sets and systems*, **86** (1997), 179–187.
- [3] DENG TING-QUAN, H. HEIJMANS. Grey-scale morphology based on fuzzy logic. CWI Report PNA-R0012, Amsterdam, October 2000.
- [4] DE BAETS B. Fuzzy morphology: a logical approach. In: *Uncertainty Analysis in Engineering and the Sciences* (Eds B. Ayyub, M. M. Gupta), Kluwer Academic Publishers, Norwell, 1997.
- [5] HEIJMANS H. *Morphological image operators*. Academic Press, Boston, 1994.
- [6] KAUFMANN A., M. M. GUPTA. *Introduction to fuzzy arithmetic: theory and applications*. Van Nostrand Reinhold, NY, 1985.
- [7] MARKOV S. M. Extended interval arithmetic. *C. R. Acad. Bulg. Sci.*, **30** (1977), No 9, 1239–1242.
- [8] NGUYEN H. T., E. A. WALKER. *A first course in fuzzy logic* (2nd edition). CRC Press, Boca Raton FL, 2000.

- [9] POPOV A. T. A relation between morphological and interval operations. *Reliable computing*, **4** (1998), No 2, 167–178.
- [10] POPOV A. T. General definition of fuzzy mathematical morphology operations. Image and non-image applications. In: *Studies in Fuzziness and Soft Computing*, **210** *Soft Computing in Image Processing - Recent Advances*, chapter 13 (Eds M. Nachtgael, D. Van der Weken, E. E. Kerre, W. Philips), Springer-Verlag, Berlin, 2007.
- [11] SERRA J. *Image analysis and mathematical morphology*, Academic Press, London 1982.
- [12] SERRA J. *Mathematical morphology for complete lattices*. In: *Image analysis and mathematical morphology* (Ed. J. Serra), **2**, Academic Press, London, 1988.
- [13] SOILLE P. *Morphological image analysis* (2nd edition), Springer-Verlag, Berlin, 2002.

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