DEVELOPMENT AND IMPLEMENTATION OF NURBS MODELS OF QUADRATIC CURVES AND SURFACES[*]

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ABSTRACT. This article goes into the development of NURBS models of quadratic curves and surfaces. Curves and surfaces which could be represented by one general equation (one for the curves and one for the surfaces) are addressed. The research examines the curves: ellipse, parabola and hyperbola, the surfaces: ellipsoid, paraboloid, hyperboloid, double hyperboloid, hyperbolic paraboloid and cone, and the cylinders: elliptic, parabolic and hyperbolic. Many real objects which have to be modeled in 3D applications possess specific features. Because of this these geometric objects have been chosen. Using the NURBS models presented here, specialized software modules (plug-ins) have been developed for a 3D graphic system. An analysis of their implementation and the primitives they create has been performed.

1. Introduction. The development of models appropriate for design in 3D graphic applications is a main topic of Computer-Aided Geometric Design (CAGD), which is a basic field of Computer Graphics. CAGD is concerned with finding mathematical methods and algorithms for representation of geometric forms with curvilinear boundaries [3].

There are many models for representing curves and surfaces in CAGD [2, 16] but for practical purposes they do not always suffice. This is because consumers' tasks and requirements continuously increase. This consumers' approach constantly makes the development of new geometric models a topical goal. The basic mathematical models of curves and surfaces that are used most in graphic applications for design are Bezier, Spline, B-spline and NURBS (Non-Uniform Rational B-Spline) [4, 6, 7, 15, 17, 18].

The main expectation towards 3D graphic applications is that they should have geometric models and tools improving representation of real objects and phenomena [1]. Because of this a group of objects having specific and important features have been investigated. These are the curves and surfaces which could be represented by one general equation (one for the curves and one for the surfaces). The curves are: ellipse, parabola and hyperbola; the surfaces are: ellipsoid, paraboloid, hyperboloid, double hyperboloid, hyperbolic paraboloid, cone; and the cylinders: elliptic, parabolic and hyperbolic. The goal of this research is to offer useful models, effective algorithms and techniques for their implementation as graphic tools in 3D graphic systems.

This representation is not so developed with regard to the widespread NURBS models of curves and surfaces in contemporary 3D graphic systems.
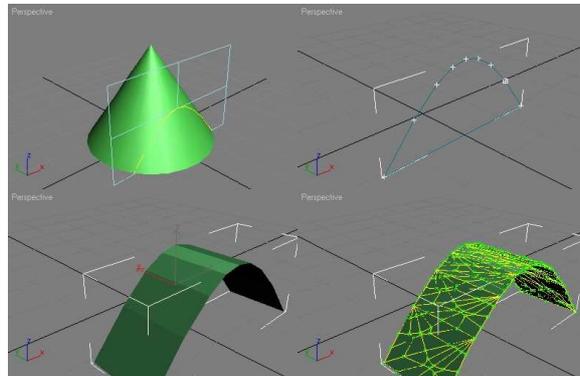

Fig. 1. Creation of a hyperbolic cylinder as a NURBS
surface when standard techniques are applied

Furthermore, most of the quadratic surfaces do not exist as graphic tools in these applications.

3D graphic systems have tools for creating ellipses and the following quadratic surfaces: sphere, cone and circular cylinder. In some applications these objects are represented parametrically, so they cannot be modified; in others (for example: Autodesk Maya) there is a NURBS representation. In those applications where NURBS representation does not exist, the surfaces must be converted into NURBS (if the system has such a possibility). But this conversion is a very expensive process with regard to the resources of the computer system.

There are some applications that offer techniques for building almost all curves and surfaces mentioned above but the creation is very slow and complicated. For example: if a patch of a hyperbolic cylinder is needed, then an arc of a hyperbola is built by an intersection of a circular cone with a plane. By its extruding, a surface is obtained. Then the surface is reorganized to *Editable Patch* and then to *NURBS surface* (Fig. 1). In this way, the generators[1] are created by several techniques, the steps for building the desired surfaces are many and the NURBS surfaces have many control vertices, which makes modeling difficult.

Thus the final goal of this research is for NURBS models of the quadratic curves and surfaces to be developed and for these models to be appropriate for development of software tools (plug-ins) that represent graphic primitives for the existing 3D *graphic systems.*

The quadratic surfaces have been investigated with regard to their NURBS representation and have been separated in three groups:
– Rotationally constructed: ellipsoid (E), paraboloid (P), hyperboloid (H), double hyperboloid (DH) and cone (C);
– Cylinders: elliptic cylinder (EC), parabolic cylinder (PC), hyperbolic cylinder (HC);
– Hyperbolic paraboloid (HP).
  Several requirements have emerged with regard to this representation:
• Curves and surfaces will be given by their parametric equations.
• The new models that are developed as a part of this research should allow local modification.
• The new NURBS models should be appropriate for the development of graphic tools.

2. Nurbs curves and surfaces. To specify a B-spline function, a special set of knots (*knot-vector*) is chosen. Every curve approximates points called

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[1]Curves through which surfaces are build.

control vertices which form a *control polygon* (Fig. 2). In [23] a definition of a *B-spline* function of degree r is given but the most often used are the *normalized B-spline* functions [15]:

(1)
$$N_{i,r}(t) = \frac{t - t_i}{t_{i+r} - t_i} N_{i,r-1}(t) + \frac{t_{i+r+1} - t}{t_{i+r+1} - t_{i+1}} N_{i+1,r-1}(t),$$

$$N_{i,0}(t) = \begin{cases} 1 & t_i \leq t \leq t_{i+1} \\ 0 & t \notin [t_i, t_{i+1}] \end{cases}.$$

Here is the parametric equation of a rational B-spline curve

(2)
$$\gamma(t) = \frac{\sum\limits_{i=0}^{n-1} P_i w_i N_{i,r}(t)}{\sum\limits_{j=0}^{n-1} w_j N_{j,r}(t)},$$

where P_i are the control vertices with weights w_i and t_j are $m + 1$ knots. If the knot-vector is $T = \left\{ \underbrace{0, \ldots, 0}_{r+1}, t_{r+1}, \ldots, t_{m-r-1}, \underbrace{1, \ldots, 1}_{r+1} \right\}$, then $\gamma(t)$ passes trough the first and the last control points.

2.1. General model of a NURBS curve. A NURBS curve of degree r is a rational B-spline curve of degree r specified by Eq. (2) but defined upon a non-uniform knot-vector (Fig. 2). Definitions, all characteristics and the geometric properties of the NURBS curves can be seen in [15]. Introducing the piecewise
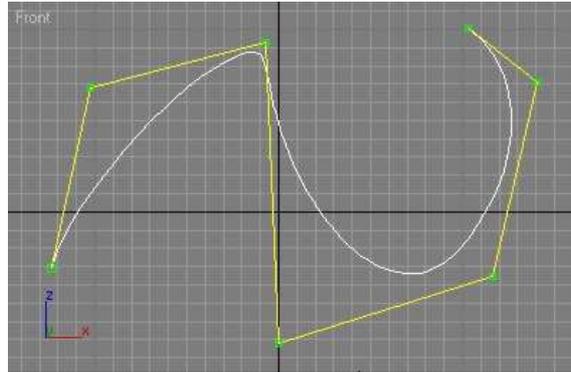

Fig. 2. A NURBS curve ($w_i = 1.0$, $i = 0, \ldots, 6$, $i \neq 2$; $w_2 = 9.0$)

rational basis functions

$$(3) \qquad R_i^r(t) = \frac{N_{i,r}(t)w_i}{\sum\limits_{j=0}^{n-1} w_j N_{j,r}(t)}$$

the curve can be written as

$$(4) \qquad \gamma(t) = \sum_{i=0}^{n-1} R_i^r(t) P_i.$$

2.2. General model of a NURBS surface. A NURBS surface (Fig. 3) of degree p in direction u and degree q in direction v is a bi-variate vector-valued piecewise rational function [15] of the form

$$(5) \qquad S(u,v) = \frac{\sum\limits_{i=0}^{n} \sum\limits_{j=0}^{m} N_{i,p}(u) N_{j,q}(v) w_{i,j} P_{i,j}}{\sum\limits_{i=0}^{n} \sum\limits_{j=0}^{m} N_{i,p}(u) N_{j,q}(v) w_{i,j}}, \qquad 0 \le u, v \le 1.$$
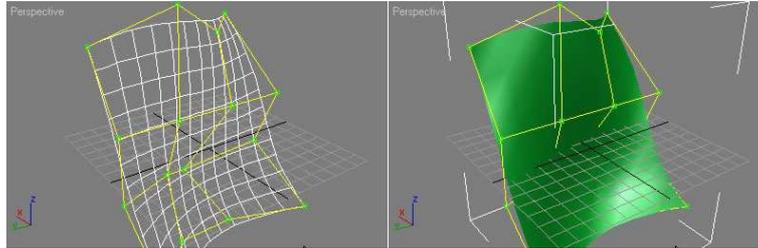

Fig. 3. A NURBS surface

The $\{P_{i,j}\}$ form the bi-directional control net, the $\{w_{i,j}\}$ are the weights and the $\{\{N_{i,p}(u)\}, \{N_{j,q}(v)\}\}$ are the non-rational B-spline basis functions defined on the knot vectors $U = \left\{ \underbrace{0, \ldots, 0}_{p+1}, u_{p+1}, \ldots, u_{r-p-1}, \underbrace{1, \ldots, 1}_{p+1} \right\}$ and $V = \left\{ \underbrace{0, \ldots, 0}_{q+1}, v_{q+1}, \ldots, v_{s-q-1}, \underbrace{1, \ldots, 1}_{q+1} \right\}$ where $p = n+p+1$ and $s = m+q+1$.

Introducing the piecewise rational basis functions

$$(6) \qquad R_{i,j}^{p,q}(u,v) = \frac{N_{i,p}(u)N_{j,q}(v)w_{i,j}}{\sum\limits_{k=0}^{n}\sum\limits_{l=0}^{m}N_{k,p}(u)N_{l,q}(v)w_{k,l}}$$

the surface can be written as

$$(7) \qquad S(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m}R_{i,j}^{p,q}(u,v)P_{i,j}.$$

3. Development of nurbs models of the quadratic curves.

Defining an arc of a quadratic curve (ellipse (ε), parabola (π) or hyperbola (χ)) is done by the values of the parameters from its parametric equations. More on the quadratic curves can be seen in [21, 24]. The curves are examined in an orthogonal coordinate system $K = \{O, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ in $\mathbb{R}^3$. This is because they have to be implemented in a 3D graphic system as graphic primitives.

3.1. The NURBS representation. The NURBS representation of a curve is made when the positions of the control vertices and their weights are determined in the NURBS model. Thus a NURBS curve can be easily subdivided. This provides a possibility for modification of different parts of it. When some parts are changed the others preserve the form of the quadratic curve and their specific features.

*3.1.1. Representing an arc of an ellipse by a NURBS curve.*A continuous arc γ_ε (Fig. 4) from ε is represented in [11] by a quadratic NURBS curve γ_ε^N. It is done when a unit-circle arc c is represented by a NURBS curve first. γ_ε is
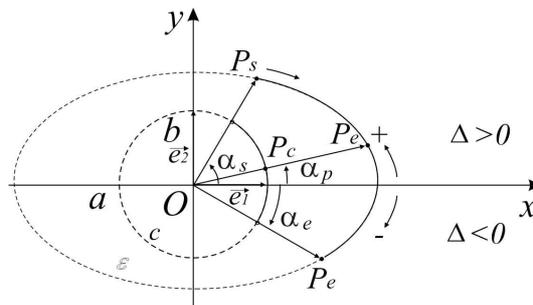

Fig. 4. Representing an arc of an ellipse by a NURBS curve

determined by a, b, α_s and α_e. γ_ε^N is of a type of the curve from Eq. (8). The parametric equation contains the requirement $\alpha \in [0; 2\pi]$ but in this research α is allowed to vary in the range $[-mf; +mf]$, mf-max float.

$$(8) \qquad \gamma(t) = \sum_{i=0}^{n-1} R_i^2(t) P_i.$$

The weights of the control vertices P_i and the values of the knots-vector which determine this curve can be seen in [11]. Some examples of ellipse-arcs can be seen in Figure 5.
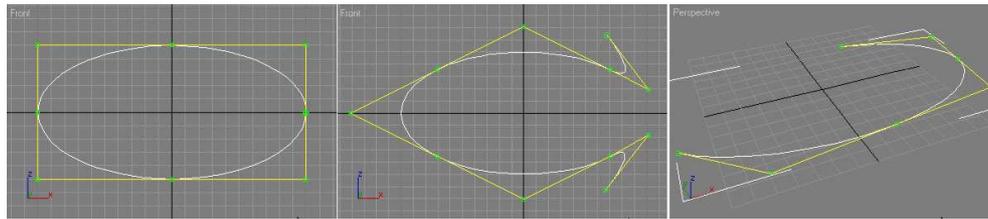

Fig. 5. A full NURBS ellipse; an arc from $-45°$ to $405°$ and the positions of the end points were changed; an arc from $210°$ to $430°$

3.1.2. Representing an arc of a parabola by a NURBS curve. Let u_s determine the point $P_s(x_s, y_s, 0)$ (start point) and u_e determine the point $P_e(x_e, y_e, 0)$ (end point) from π (Fig. 6). Thus varying the parameter u from u_s to u_e in the
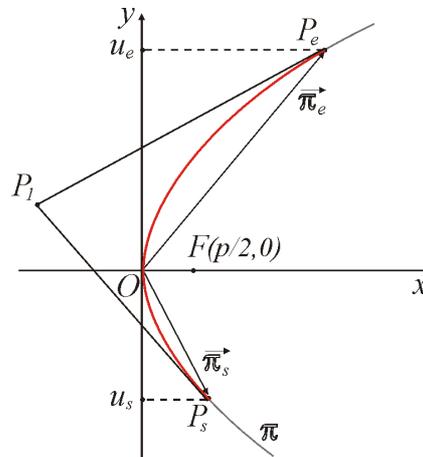

Fig. 6. Representing an arc of parabola by a NURBS curve

parametric equation of the curve an arc of π is obtained. Let this arc be δ_π. In [10] a proposition is given which proves that the graphics of δ_π is identical with the graphics of a NURBS curve β specified by Equation (4) for $r = 2$, a knot-vector of the following type $T = \{0, 0, 0, 1, 1, 1\}$ and control points: $P_s = (u_s^2, 2pu_s)$, $P_1 = (u_e((u_e - u_s)^2(u_e + u_s) - u_e),\ p(u_e - u_s)^2(u_e + u_s))$, $P_e = (u_e^2, 2pu_e)$ with weights $w_0 = w_1 = w_2 = 2p$.

Then β is reorganized to a NURBS curve γ_π^N (such a technique is given in [10]). γ_π^N is defined upon a knot-vector of type $T = \{0, 0, 0, t_1, t_1, t_1, \ldots, t_{m-1}, t_{m-1}, t_{m-1}, 1, 1, 1\}$ and control points P_j, $j = 0, \ldots, 2m$. Thus γ_π^N contains m parts that we can modify locally. NURBS parabolas are shown in Figure 7.
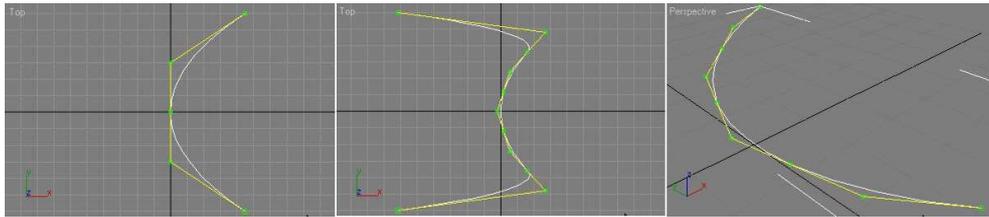

Fig. 7. A NURBS parabola ($p = 40$, $u \in [-60; 60]$); the positions of the end points were changed; an arc for $u \in [-30; 60]$

3.1.3. Representing an arc of a hyperbola by a NURBS curve. An arbitrary arc from χ (Fig. 8) is represented in [12] by a quadratic NURBS curve γ_χ^N. Let this arc be χ_β defined by α_B and α_L (start and end angles). In [12] the following proposition is proved: If an arc χ_β from χ is given by the point B obtained for the
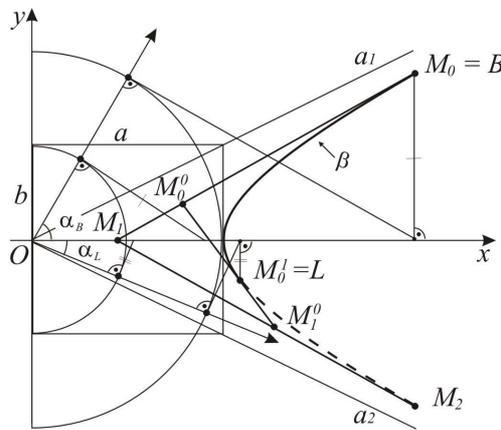

Fig. 8. Representing an arc of hyperbola by a NURBS curve

value of an angle α_B and the point L obtained for the value of an angle α_L, where $|\alpha_B| > |\alpha_L|$ and $\alpha_B, \alpha_L \in \left(\dfrac{3\pi}{2}; 2\pi\right) \cup \left[0; \dfrac{\pi}{2}\right)$ or $\alpha_B, \alpha_L \in \left(\dfrac{\pi}{2}; \dfrac{3\pi}{2}\right)$, then the graphics of χ_β and the graphics of the quadratic NURBS curve β defined upon a knot-vector of type $T = \{0, 0, 0, 1, 1, 1\}$ with control points:

$$\gamma: \begin{cases} M_0^w = (\ a,\ b.\sin\alpha_B,\ \cos\alpha_B) \\ M_0^{0w} = ((1 - t^*).a + \text{sign}(\cos\alpha_B).t^*.a.\cos\alpha_B,\ (1 - t^*).b.\sin\alpha_B,\ w.\cos\alpha_B)\,, \\ M_0^{1w} = (\ a,\ b.\sin\alpha_L,\ \cos\alpha_L) \end{cases}$$

where $w = (1 - t^*) + t^*.\left(\text{sign}(p_B)\left((1/p_B) + p_B\right) - 1\right)$, $p_B = \cos\alpha_B$, $t^* = \dfrac{p}{1 + p}$,

$p = \sqrt{\dfrac{|M_0 Q|}{|Q M_2|}}$, are identical.
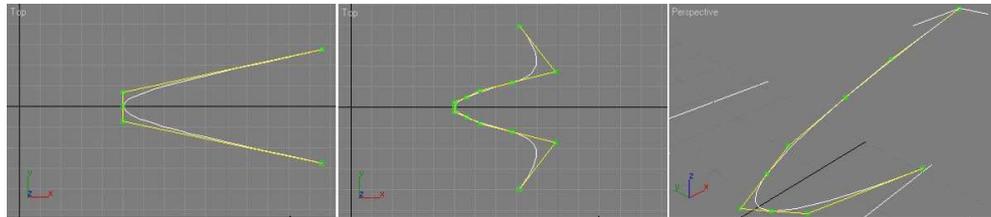

Fig. 9. A NURBS hyperbola ($a = 50$, $b = 10$, from $290°$ to $70°$); the positions of the end points were changed; an arc from $315°$ to $70°$

Then β is reorganized to a NURBS curve γ_χ^N (such a technique is given in [12]). γ_χ^N is defined upon a knot-vector of type $T = \{0, 0, 0, t_1, t_1, t_1, \ldots, t_{m-1}, t_{m-1}, t_{m-1}, 1, 1, 1\}$ and control points P_j, $j = 0, \ldots, 2m$. Thus γ_χ^N contains m parts that can be modified locally. NURBS hyperbolas are shown in Figure 9.

4. Development of NURBS models of the quadratic surfaces.

Defining a patch of a quadratic surface is done by the values of the parameters from its parametric equations.

4.1. Analytic representation of the surfaces

The quadratic surfaces discussed here are ruled and revolved. A classification and their characteristics can be seen in [24, p. 172–182]. Here only two final theorems are quoted:

Theorem 4.1. *The real revolved surfaces of second degree with rotational axes Oz are: sphere (ellipsoid at $a = b = c$); ellipsoid at $a = b$; paraboloid at $a = b$; hyperboloid at $a = b$; double hyperboloid at $a = b$; cone at $a = b$; elliptic cylinder at $a = b$.*

Theorem 4.2. *The real ruled surfaces of second degree are: cone, elliptic cylinder, parabolic cylinder, hyperbolic cylinder, hyperboloid and hyperbolic paraboloid.*

4.2. Analysis of the surfaces. The surfaces represented in conical forms are considered in a right orthogonal coordinate system $K = \{O, \vec{e_1}, \vec{e_2}, \vec{e_3}\}$ ($K = Oxyz$).

4.2.1. Building on the rotation. The surfaces E, P, H, DH and C are discussed here. Applying a method of metric classification convenient scalar-parametric equations of surfaces are obtained. In [14,15] definitions, theorems, methods of metric classifications and studies of these surfaces can be seen. The graphics of the five surfaces are shown in Figure 10.
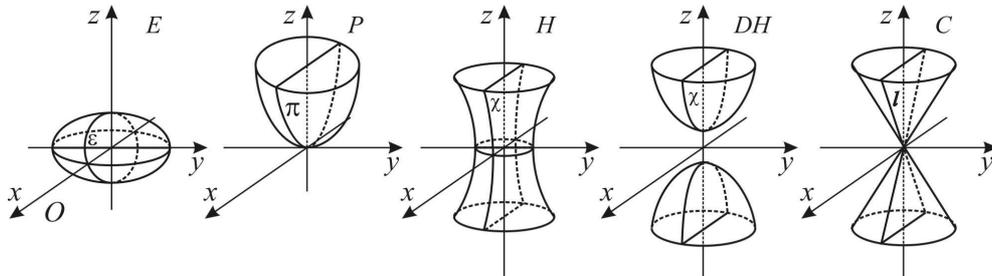

Fig. 10. E – ellipsoid; P – paraboloid; H – hyperboloid; DH – double hyperboloid; C - cone; and generators: ε – ellipse, χ – hyperbola, π – parabola, l – straight line

The intersections of these surfaces with the planes

$$\alpha_t : z = t \ (\| \ Oxy) \,, \quad \beta_p : y = p \ (\| \ Oxz) \,, \quad \gamma_q : x = q \ (\| \ Oyz)$$

give information on how the surfaces could be built. By intersecting E, P, H, DH and C (at $a = b$) with $\beta_p : y = 0$ the generators are obtained. These curves are parts of: ellipse (ε) at $E(a = b)$, hyperbola (χ) at $H(a = b)$ and $DH(a = b)$, parabola (π) at $P(a = b)$ and straight line (l) at $C(a = b)$ (Fig. 10). By intersecting any of the surfaces with α_t in these cases circles are obtained. *Therefore, each of these surfaces can be constructed by rotation of the generators around Oz.*

4.2.2. Cylindrical surfaces. The quadratic surfaces are presented analytically by one general equation [22]. When a metrical classification is made (to

put them into a standard position in an orthogonal coordinate system), the convenient parametric equations are obtained [5, 22]. In [24] definitions, theorems and methods for metrical classifications and investigations of the cylindrical surfaces (Fig. 11) can be seen.
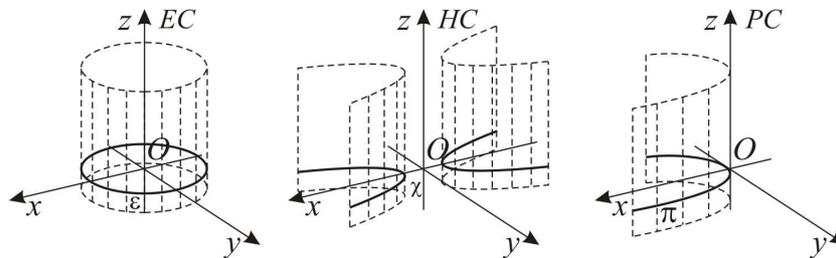

Fig. 11. The cylinders EC, HC, PC and their generators ε – ellipse, χ – hyperbola, π – parabola

The intersections of these surfaces with the planes

$$\alpha_t : z = t \ (\| \ Oxy), \quad \beta_p : y = p \ (\| \ Oxz), \quad \gamma_q : x = q \ (\| \ Oyz)$$

give information on how the surfaces could be built. By intersecting EC, HC and PC with α_t the generators which are ellipse (ε), hyperbola (χ) and parabola (π) are obtained (Fig. 11 [24]), and with β_p and γ_q straight lines which are parallel to Oz are obtained. *Therefore, by a ruled movement of the generators in a parallel to Oz direction, every one of the cylinders could be built.*

4.2.3. Hyperbolic paraboloid. The point $O = HP \cap Oxy$ is called *vertex* of the hyperbolic paraboloid. In [22] we can find a study of the sections of the HP with the plains

$$\alpha_t : z = t \ (\| \ Oxy), \quad \beta_p : y = p \ (\| \ Oxz), \quad \gamma_q : x = q \ (\| \ Oyz)$$

and draw conclusions on how they part HP. It is very important to look at how β_p and γ_q part HP.

The sections $HP \cap \beta_p : y = p$ are the parabolas

$$(9) \qquad \pi_p : x^2 = 2a^2 \left(z + \frac{p^2}{2b^2} \right)$$

which are located in β_p on the direction of $\| \ Oz^+$ and have vertices $F_p \left(0, \ -\frac{p^2}{2b^2} \right)$.

The sections $HP \cap \gamma_q : x = q$ are the parabolas

(10)
$$\pi_q : y^2 = -2b^2 \left(z - \frac{q^2}{2a^2} \right)$$

that are located in γ_q on the direction of $\parallel Oz^-$ and have vertices $G_q \left(0, \dfrac{q^2}{2a^2} \right)$.
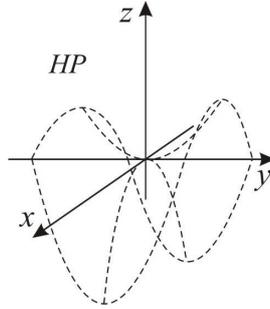

Fig. 12. Hyperbolic paraboloid – HP

4.3. The NURBS representation. For the construction of NURBS models of the surfaces the NURBS models of the curves are used. The curves are used as generators. Every NURBS model of a surface is developed by means of the parameters of its parametric equations.

4.3.1. Representing E, P, H, DH and C by bi-quadratic NURBS surfaces

The surfaces of E, P, H, DH and C can be constructed using their special cases (E at $a = b$ и $a = b = c$ (sphere (S)); P at $a = b$; H at $a = b$; DH at $a = b$ and C at $a = b$) by revolving the appropriate generators [14]. After the generators have been found they are represented as NURBS curves using the models given in Section 3.

Let $\gamma(v) = \sum\limits_{j=0}^{m} R_{j,2}(v) P_j$ be a NURBS curve of second degree defined under the knot vector V. The curve will be revolved around an axis. $\gamma(v)$ is set to lie in the Oxz plain. The curve has $n + 1$ control vertices $P_j(x_j, y_j, z_j)$ with weights w_j and a knot vector $V = \{0, 0, 0, v_1, v_1, v_2, v_2, \ldots, v_k, v_k, 1, 1, 1\}$. Now $\gamma(v)$ is revolved around the Oz axis. As a result a surface is obtained. Let this surface be $S(u, v)$. Then $S(u, v)$ must have the following features:

1) for fixed $\bar{u}$, $S(\bar{u}, v)$ is the curve $\gamma(v)$ revolved around the Oz axis;

2) for fixed $\bar{v}$, $S(u, \bar{v})$ is an arc of a circle which lies in the plane that is perpendicular to the Oz axis and its center lies on the axis.

The knot vector U is chosen from the sort $U = \{0, 0, 0, u_1, u_1, u_2, u_2, \ldots,$ $u_h, u_h, \ldots, 1, 1, 1\}$ and the weights are chosen as $w_i = \{1, \sqrt{2}/2, 1, \sqrt{2}/2, \ldots,$ $1, \sqrt{2}/2, 1\}$, for $i = 0, \ldots, n$. The weights are $w_i = \sqrt{2}/2$, $i = 1, 3, \ldots, n-1$ because $\sqrt{2}/2$ is the value of the weight of the middle control vertex in the representation of an arc of a circle. Then the surface has the form:

$$(11) \qquad S(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} R_{i,2;j,2}(u, v) P_{i,j}.$$

The knot vectors are U and V. The control vertices and the weights are determined as follows: for $i = 0$, $P_{i,j} = P_{0,j} = P_j$. All control vertices $P_{i,j}$ for a fixed j lie in the plane $z = z_j$. The weights are determined by the product of w_j with circular weights w_i, i.e. for a fixed j, $w_{0,j} = w_j$, $w_{1,j} = \sqrt{2}/2w_j$, $w_{2,j} = w_j$, $w_{3,j} = \sqrt{2}/2w_j$, $w_{4,j} = w_j$, $\ldots$, $w_{n-1,j} = \sqrt{2}/2w_j$, $w_{n,j} = w_j$.

Example: Construction of NURBS parabolic surface (NP-surface).

NP-surface problem. Construct a parabolic surface P by a bi-quadratic NURBS surface for given values of the parameters $a, b > 0$, u and v: $u \in [u_s; u_e]$, $u_s, u_e \in J \subseteq \mathrm{R}^+$, $v \in [v_s; v_e]$, $v_s, v_e \in J \subseteq \mathbb{R}$.

A generator of this surface is placed in the Oxz plane. It represents an arc of a parabola. This curve is represented by a NURBS curve of second degree using a method given in Section 3.

When a rotational construction of a bi-quadratic NURBS surface is made on the method given here a patch of a paraboloid is obtained at $a = b$. It is determined by the parameters u_s, u_e, v_s, v_e. Let the paraboloid at $b = a$ be $P_{b=a}$. $P_{b=a}$ is represented by a bi-quadratic NURBS surface from the type of Eq. 11.

Then the paraboloid P is obtained by scaling $P_{b=a}$ with a matrix of transformation $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b/a & 0 \\ 0 & 0 & 1 \end{pmatrix}$: $P = T.P_{b=a}$. The control vertices of the NURBS surface are obtained by $Q_{i,j} = T.P_{i,j}$. Its equation is given by:

$$(12) \qquad P_Q(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} R_{i,2;j,2}(u, v) Q_{i,j}.$$

Figure 13 shows two NURBS paraboloid-surfaces (NP-surfaces). They are visualized with their control meshes which consist of the control vertices $Q_{i,j}$.

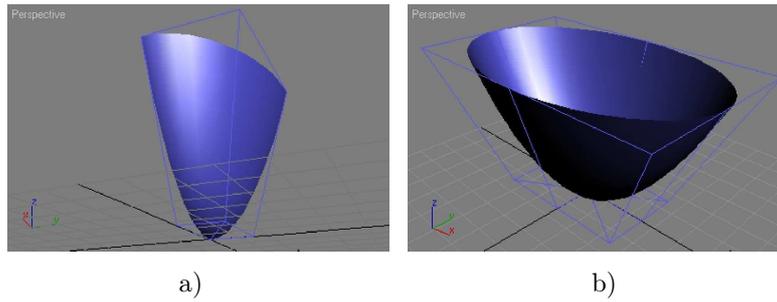The detailed construction of the other surfaces can be seen in [8, 14].

a)   b)

Fig. 13. a) NP-surface at $a = 5$, $b = 3$, $u_s = 0$, $u_e = 10$ $v_s = 0°$, $v_e = 90°$;
b) NP-surface at $a = 5$, $b = 3$, $u_s = 0$, $u_e = 10$ $v_s = 0°$, $v_e = 360°$

4.3.2. Representing EC, PC and HC by bi-quadratic NURBS surfaces. As was mentioned above, every surface of a cylinder is a ruled surface. Therefore a construction of a ruled bi-quadratic NURBS surface is given here as well.

Each one of the generators is presented as a NURBS curve that lies in the plane Oxy and has $n+1$ control vertices $P_i(x_i, y_i, z_i)$, weights w_i and a knot-vector $U = \{0, 0, 0, u_1, u_1, u_2, u_2, \ldots, u_k, u_k, 1, 1, 1\}$.

A patch of a cylinder is specified by the parameters of the parametric equations for this surface and two parameters more are needed to set up the beginning and the end of the surface along z. Let these parameters be t_s and t_e (t_s, $t_e \in \mathrm{R}$). Let $\gamma(u) = \sum\limits_{i=0}^{n} R_i^2(u) P_i$ be the generator (NURBS curve in Oxy obtained by one of the methods mentioned above) of the surface on the knot vector U and with weights w_i (Fig. 14a). Let $\vec{Q} = (0, 0, 1)$ if $t_s < t_e$, and $\vec{Q} = (0, 0, -1)$ if $t_s > t_e$, and $d = |t_s - t_e|$. Then $P_i.z = t_s$, $i = 0, \ldots, n$, setting the curve in the start position (Fig. 14b). Let $S(u, v)$ be the required surface obtained by sweeping $\gamma(u)$ to a distance d along $\vec{Q}$. Denoting the parameter for the sweep direction by v, $0 \leq v \leq 1$, clearly $S(u, v)$ must satisfy two conditions:

1) for fixed $\bar{u}$, $S(\bar{u}, v)$ is a straight line from $\gamma(\bar{u})$ to $\gamma(\bar{u}) + d\vec{Q}$;

b) for fixed $\bar{v}$, $S(u, \bar{v}) = \gamma(u) + \bar{v}d\vec{Q} = \sum\limits_{i=0}^{n} R_i^2\left(P_i + \bar{v}d\vec{Q}\right)$. The desired representation is $S(u, v) = \sum\limits_{i=0}^{n} \sum\limits_{j=o}^{2} R_{i,j}^{2,2}(u, v) P_{i,j}$.

The knot vectors are U and V, where $V = \{v_0 = 0, 0, 0, v_1, v_1, \ldots, v_{h-1}, v_{h-1}, v_h = 1, 1, 1\}$, h is the number of the segments, v_l are chosen as follows: $f = 1/h$, $v_0 = 0$ and $v_{l+1} = v_l + f$, for $l = 0, \ldots, h - 1$, and U is the knot vector
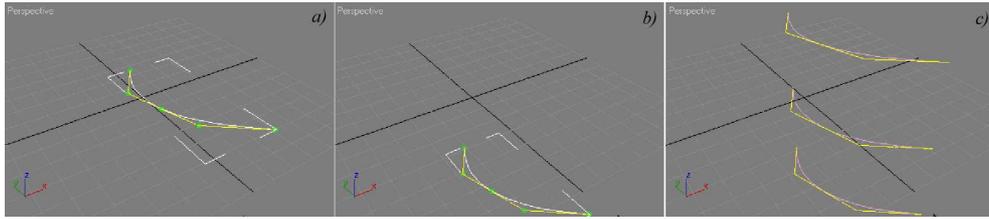
Fig. 14. A generator as a NURBS curve

of $\gamma(u)$. The control vertices are $P_{i,0} = P_i$ and $P_{i,j}$, $j = 1, \ldots, m$, determined as follows: $x_{i,j} = x_{i,0}$, $y_{i,j} = y_{i,0}$, $z_{i,j} = z_{i,0} + j.q$, where $q = d/2h$. The weights are $w_{i,j} = w_i$.
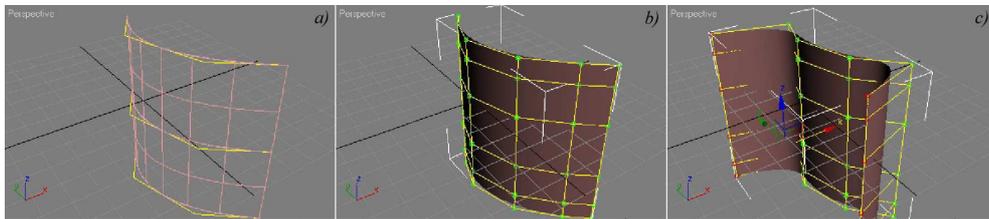

Fig. 15. A cylinder as a NURBS surface.

A bi-quadratic NURBS surface is constructed to represent the patch of a cylinder. The final result can be seen in Figure 15 where a local modification of the NURBS surface has been made as well.

In Figure 16 NURBS Parabolic Cylinders (NPC-surfaces) can be seen.

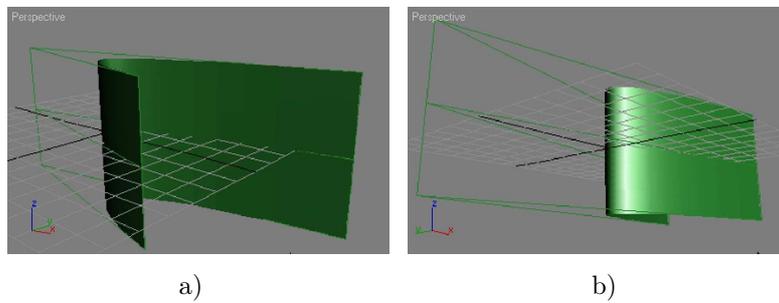a)   b)

Fig. 16. a) NPC-surface at $p = 5$, $u_s = -10$, $u_e = 30$, $t_s = -30$, $t_e = 30$;
b) NPC-surface at $p = 5$, $u_s = -30$, $u_e = 30$, $t_s = -30$, $t_e = 30$

In [13] a detailed construction of the cylindrical surfaces can be seen.

4.3.3. Representing HP by a bi-quadratic NURBS surface. **NHP-surface Problem:** Construct a surface of HP by a bi-quadratic NURBS surface for given values of the parameters $a, b > 0$.

A hyperbolic paraboloid is a surface symmetrical with respect to the planes Oxz and Oyz. In [24] it can be seen that at $y = p$ ($HP \cap \beta_p : y = p$) all points of HP form the parabolas π_p. Then the values of x can be placed in the range $x \in [x_s, x_e]$. At $x = q$ ($HP \cap \gamma_q : x = q$) all points of HP form the parabolas π_q. Then the values of y can be placed in the range $y \in [y_s, y_e]$.

A hyperbolic paraboloid is delimited by the planes $HP \cap \gamma_{xs} : x = x_s$, $HP \cap \gamma_{xe} : x = x_e$, $HP \cap \beta_{ys} : y = y_s$, $HP \cap \beta_{ye} : y = y_e$. The two conditions $x_s = -x_e$ and $y_s = -y_e$ determine completely the HP surface and ensue symmetry with respect to the Oxz and Oyz planes. HP is created for $x \in [x_s, x_e]$ and $y \in [y_s, y_e]$. Thus the surface HP_d which is derived from HP is built of parabolic curves in the directions of x and y. It follows from this that HP_d can be built as a bi-quadratic NURBS surface which is given with the following equation:

$$(13) \qquad\qquad HP_d(u, v) = \sum_{i=0}^{2} \sum_{j=0}^{2} R_{i,2,j,2}(u, v)\, P_{i,j}$$

HP_d is defined under the knot-vectors $U = \{0, 0, 0, 1, 1, 1\}$ and $V = \{0, 0, 0, 1, 1, 1\}$, substituting direction x for direction u, and direction y for direction v in the NURBS basis.

The control vertices $P_{0,0}, P_{0,2}, P_{2,0}, P_{2,2}$ for the given values of x_s, x_e, y_s and y_e are obtained. The control vertices $P_{0,1}, P_{1,0}, P_{1,2}, P_{2,1}$ are intersection points of the parabolas $\pi_{p=y_s}$, $\pi_{p=y_e}$, $\pi_{q=x_s}$ and $\pi_{q=x_e}$. HP goes through the point $O(0, 0, 0)$ and it is symmetrical with respect to this point. It follows from this that HP_d goes through the point O at $u = v = 1/2$. Then the control vertex $P_{1,1}(x_{1,1}, y_{1,1}, z_{1,1})$ is determined by $S(1/2, 1/2) = (0, 0, 0)$ where $x_{1,1}, y_{1,1}, z_{1,1}$ are only unknown quantities.

Here are the results for the control vertices $P_{ij}(x_{ij}, y_{ij}, z_{ij})$, $i, j = 0, 1, 2$ which determine the NURBS surface:

$P_{00}(x_s,\ y_s,\ dx - dy),\quad P_{01}(x_s,\ 0,\ dx + dy),\qquad\qquad P_{02}(x_s,\ y_e,\ dx - dy)$

$P_{10}(0,\ y_s,\ -dx - dy),\quad P_{11}\left(0,\ 0,\ -\sum \Big/ \left(N_{1,2}\left(\tfrac{1}{2}\right)\right)^2\right),\quad P_{12}(0,\ y_e,\ -dx - dy),$

$P_{20}(x_e,\ y_s,\ dx - dy),\quad P_{21}(x_e,\ 0, dx +\ dy),\qquad\qquad P_{22}(x_e,\ y_e,\ dx - dy),$

where $dx = \dfrac{x_s^2}{2a^2}$, $dy = \dfrac{y_s^2}{2b^2}$ and

$$\sum = \; N_{0,2}.\left(N_{0,2}.z_{00} + N_{1,2}.z_{01} + N_{2,2}.z_{02}\right) + N_{1,2}.\left(N_{0,2}.z_{10} + N_{2,2}.z_{12}\right) +$$
$$N_{2,2}.\left(N_{0,2}.z_{20} + N_{1,2}.z_{21} + N_{2,2}.z_{22}\right).$$

The shape of HP is determined by the parameters a and b from its parametric equation. Thus it is constructed here by giving it a size: width (W) in the direction of x and length (L) in the direction of y. Then $x_s = -\dfrac{W}{2}$, $x_e = \dfrac{W}{2}$, $y_s = \dfrac{L}{2}$ and $y_e = -\dfrac{L}{2}$. Thus a NURBS hyperbolic paraboloid surface (NHP-surface) is constructed by giving the parameters a, b, W and L.

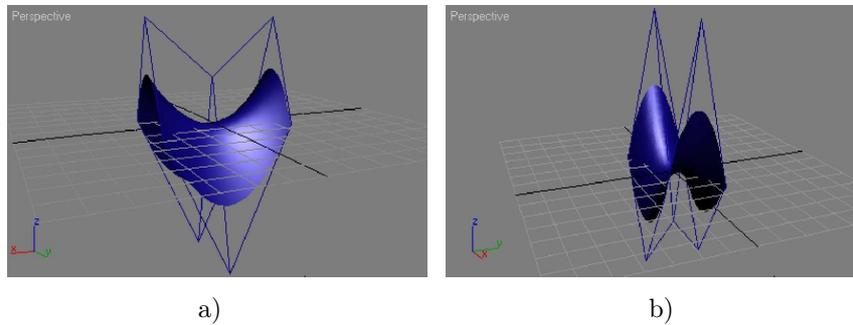Figure 17 shows two NHP-surfaces which are obtained for different values of the parameters.

a) b)

Fig. 17. a) NHP-surface at $a = 2$, $b = 2$, $W = 50$, $L = 30$;
b) NHP-surface at $a = 2.8$, $b = 2.1$, $W = 47$, $L = 28$

A subdivision is made in order to enable local modification of NHP. In [8] an approach for subdivision of bi-quadratic NURBS surfaces is proposed.

5. Implementation of the developed nurbs models of the quadratic curves and surfaces in a 3D graphic system. The efficiency, reliability and functionality of the developed models and algorithms can be demonstrated and evaluated in practice only if they are implemented in a 3D graphic system for design. This is done through the development of software modules (plug-ins) for a particular application. An analysis of their implementation and the primitives they create has been made as well.

The tools for developing new features for the 3D graphic systems have been investigated. Requirements have been set up with regard to the 3D graphic system in which the new NURBS models might be implemented. A comparative analysis

of 3D graphic systems for design has been utilized. Following that, Autodesk 3ds Max [20] has been highlighted as a 3D graphic system for the implementation of the NURBS models. This graphic application has a software development kit (SDK) and script language (MaxScript) [19] for developing new specialized software modules (SSM) which can represent new graphic primitives in the system.

5.1. Development of the SSM. For the implementation of the new NURBS models in the SSM a number of routines have been developed [8]. There are functions which evaluate the main control vertices, knot vectors and all other control vertices of the given curve or surface. There are routines that create the curve or the surface in the scene as well.

The specialized software modules contain a mechanism for collecting data from the mouse. This is used for the interactive creation of the new NURBS primitives [8].

3ds Max provides seven main views (projections) of the coordinate space. This requires the creation of NURBS curves and surfaces to be made in any one of them. This is done by determining in which view the user is modeling at that time. After that the SSM create the curves and surfaces in the selected view. The local coordinate system of the primitive (in which it must be created) is set by the rule: the axis x points to the right, y to the top and z to the front (straight to the observer).

The values of the parameters that define the arcs and the surfaces of the geometric objects are delimited by boundary values given by their parametric equations. In the implementation of the NURBS models and in the actual construction of the NURBS primitives additional boundary values are set [8] for which the SSM work correctly and objects are built accurately. Mainly that derives from the specific features of the graphic system with regard to setting numbers and dimensions in it.

5.2. Comparative analysis and demonstration of the work of SSM. A comparative analysis has been made [8]. It includes a comparison between the best possible techniques for constructing the NURBS curves and surfaces that the system has and the work of the SSM and the primitives they create. The following features have been scrutinized:

- Number of steps for the construction;

- Time for the construction;

- Possibilities for local modification of the NURBS primitives;

- Number of the control vertices;

- Visualization quality of the NURBS surfaces.

Example 1. *Creation of a NURBS parabolic curve.*

The approach of finding a conic section of the given form should be followed in order to obtain an arc of parabola [10]. In this case the curve will be a spline curve. It must then be converted to a NURBS curve. The result (Fig. 18a) is obtained in 3 steps during the construction, in about 75 seconds. It is a NURBS curve with 37 control vertices.

Figure 18b shows a NURBS parabolic curve created by the SSM NURBS Parabolic Arc in: steps—1, time—5 seconds (15 times faster than the method mentioned above) and subdivided into 2 segments, 5 control vertices. Furthermore, the SSM can create curves for different values of u_s and u (Fig. 19a). The curves can also be easily modified locally (Fig. 19b).
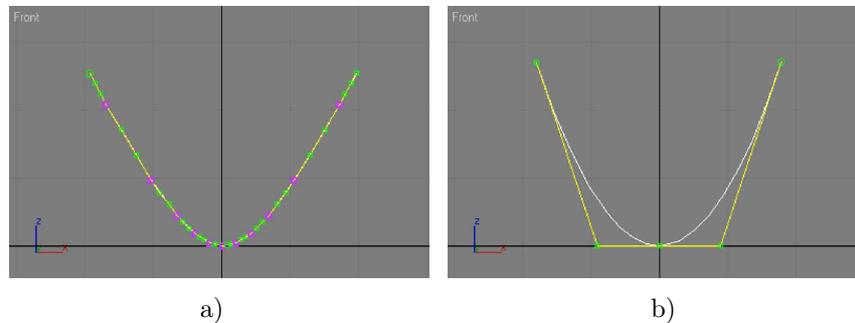

a)   b)

Fig. 18. a) Parabola obtained by cut and converted into NURBS;
b) NURBS parabola plug-ins built with NURBS Parabolic Arc of values: $p = 6$,
$u_s = -18$, $u_e = 18$, segments $= 2$

Example 2. *Creation of a NURBS hyperbolic surface.*

The construction of a hyperbolic surface as a NURBS surface can be made after a spline curve of hyperbola is obtained in Autodesk 3ds Max. The operations are parallel to those above for obtaining a parabola. The result is obtained in 6 steps during the construction, in about 240 seconds. The NURBS surface is a bi-cubic with 4864 control vertices. Figure 20a shows the NURBS hyperbolic surface, which is not sufficiently smooth. Because of the huge number of control vertices the modeling of this surface will be very difficult.

Figure 20b shows a bi-cubic NURBS surface created by the SSM *NURBS Hyperboloid* in: steps – 1, time – 5 seconds (48 times faster than the method mentioned above). If the surface is created as a bi-quadratic without subdivision,
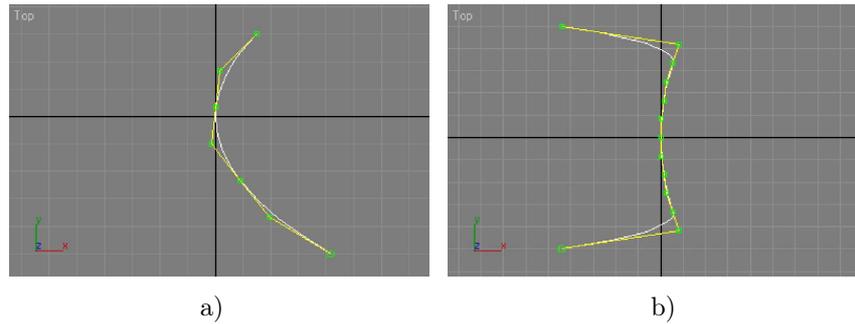
a)   b)

Fig. 19. NURBS plug-in parabola with a NURBS Parabolic Arc of values:
a) $p = 30$, $u_s = -50$, $u_e = 30$, segments $= 3$;
b) $p = 100$, $u_s = -50$, $u_e = 50$, segments $= 6$ and is locally modified

the control vertices will be 27 (around 180 times fewer). Figure 20b shows that the surface is smooth enough and the visualization is excellent.

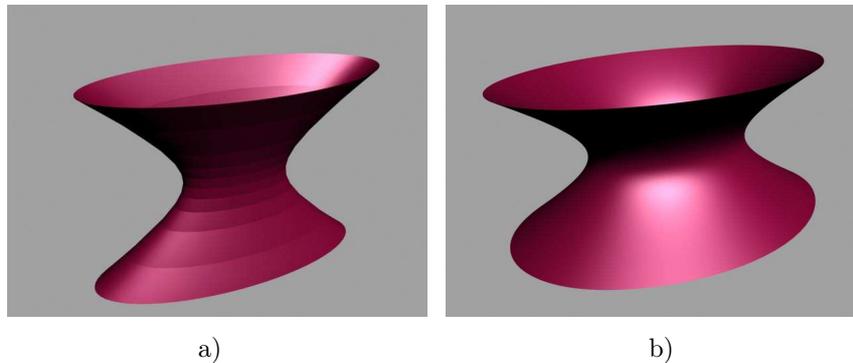a)   b)

Fig. 20. a) The bi-cubic NURBS hyperbolic surface with 4864 control vertices;
b) NURBS hyperboloid that has been created by the NURBS Hyperboloid plug-in for the following values of the parameters: $a = 40$, $b = 30$, $c = 20$, $\alpha_s = 300°$, $\alpha_e = 60°$, $\beta_s = 0°$, $\beta_e = 360°$, segments$(U, V) = 4$

Figure 21 shows two images of a) a bi-quadratic and b) a bi-cubic NURBS hyperboloids which are locally modified.

Example 3. *Creation of a NURBS hyperbolic paraboloid.*

A tool or an approach for construction of the surface of a hyperbolic paraboloid has not been found.

Figure 22 shows two images of a) a bi-quadratic and b) a bi-cubic NURBS hyperbolic paraboloid created by the NURBS Hyperbolic Paraboloid plug-in.
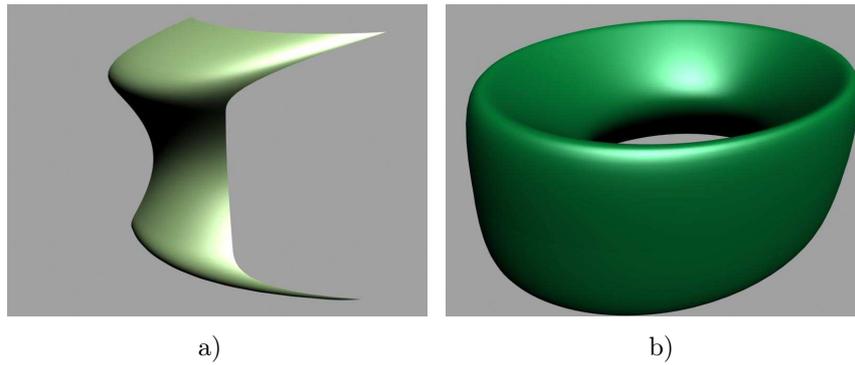
a)   b)

Fig. 21. NURBS hyperboloids that have been created by the NURBS Hyperboloid plug-in for the following values of the parameters: a) $a = 40$, $b = 40$, $c = 40$, $\alpha_s = 300°$, $\alpha_e = 60°$ $\beta_s = 0°$, $\beta_e = 75°$, segments$(U, V) = 4$; b) $a = 40$, $b = 30$, $c = 20$, $\alpha_s = 295°$, $\alpha_e = 65°$ $\beta_s = 0°$, $\beta_e = 360°$, segments$(U, V) = 8$

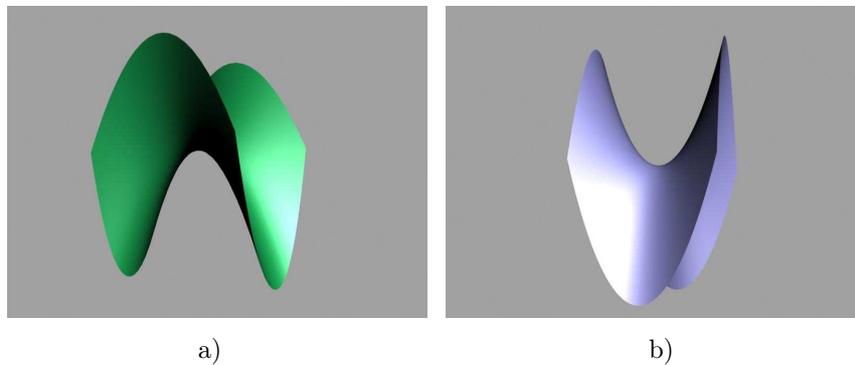a)   b)

Fig. 22. NURBS hyperbolic paraboloids that have been created by the NURBS Hyperbolic Paraboloid plug-in for the following values of the parameters: a) $a = 3$, $b = 2$, $W = 50$, $L = 30$; segments$(U, V) = 1$;b) $a = 2$, $b = 2$, $W = 50$, $L = 30$; segments$(U, V) = 4$

6. Conclusion. The presented models, algorithms and specialized software modules (plug-ins) in this article allow the creation of quadratic curves and surfaces as NURBS curves and surfaces with the following benefits:

- Number of steps in the creation process: 1 (where other approaches require a minimum of 3 for curves and 5 or 6 for surface);

- Creation in *real time*—in about 5 seconds (which is from 9 to 48 times faster than other approaches);

- Excellent opportunity for local modification;

- Minimum number of control vertices: 3 for curves and 9 for surface, while the other approaches achieve from 31 to 73 for curves and from 256 to 4864 (and more) for surface;

- High level of visualization after rendering (for surfaces).

Using these NURBS models of the quadratic curves and surfaces twelve plug-ins for Autodesk 3ds Max 3D graphic system for design have been developed [8]. They have been approved and adopted by "*HighEnd3D*" – Internet site for high technologies and 3D applications. They are also available for download as follows:

NURBS Conical Arcs 1.4 – `http://www.highend3d.com/f/4368.html`,

NURBS Quadratic Surfaces 1.2 – `http://www.highend3d.com/f/4369.html`.

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