OPERATIONAL METHODS IN THE ENVIRONMENT OF A COMPUTER ALGEBRA SYSTEM[*]

Margarita Spiridonova

*Dedicated to the 75th anniversary of Professor Ivan Dimovski,
Corresponding member of the Bulgarian Academy of Sciences.*

ABSTRACT. The presented research is related to the operational calculus
approach and its representative applications. Operational methods are con-
sidered, as well as their program implementation using the computer algebra
system Mathematica. The Heaviside algorithm for solving Cauchy's prob-
lems for linear ordinary differential equations with constant coefficients is
considered in the context of the Heaviside-Mikusiński operational calculus.
The program implementation of the algorithm is described and illustrative
examples are given. An extension of the Heaviside algorithm, developed

by I. Dimovski and S. Grozdev, is used for finding periodic solutions of linear ordinary differential equations with constant coefficients both in the non-resonance and in the resonance cases. The features of its program implementation are described and examples are given. An operational method for solving local and nonlocal boundary value problems for some equations of the mathematical physics (the heat equation, the wave equation and the equation of a free supported beam) is developed and the capabilities of the corresponding program packages for solving those problems are described. A comparison with other methods for solving the same types of problems is included and the advantages of the operational methods are marked.

The paper presents the principal results of the author's doctoral thesis [29]. Since the most important common feature of the considered operational calculi is their immediate approach to finding solutions of initial and boundary value problems, in [29] they are called direct operational methods.

1. Introduction. The main idea of operational calculus consists in transformation of calculus problems to algebraic problems, treating the differentiation operator as an algebraic object.

Some ideas of "symbolic" operational calculus come from the works of Leibnitz, Euler, Cauchy and other mathematicians (see [30], [26], and also [31]). Nevertheless, it is Oliver Heaviside (1850–1925) who is regarded as the father of operational calculus. He was the first who successfully applied this method in his research for solving initial value problems related to electromagnetic theory (see [18]). But Heaviside did not established a sound mathematical theory and his calculus was regarded by some scientists as inconsistent. The first justification of his approach was done by means of the Laplace transformation. Much later – in the middle of the last century – the Polish mathematician Jan Mikusiński (1913–1987) made a return to the original operator viewpoint and developed a direct algebraic approach to the Heaviside Operational calculus. He based his calculus on the notion of convolution quotient, without referring it to the Laplace transformation. His calculus is known as Mikusiński's operational calculus. From an historical point of view it is fair to call it the operational calculus of Heaviside–Mikusiński.

Scientists in many countries have published works related to Mikusiński's operational calculus. Some of them are L. Berg, T. K. Boehme, I. H. Dimovski, V. A. Ditkin, A. P. Prudnikov, K. Yosida, etc. Other names are mentioned in some references, for example in [26]. Some recent results can be found in [34], [35], [24] and others.

In the presented work, mainly, results by I. H. Dimovski on the development of operational calculi of Mikusiński's type are used.

Operational calculus has been widely used for solving problems in mathematics, physics, mechanics, electrical engineering, etc. The algorithms and the program tools described in this paper are intended to facilitate the use of the operational calculus approach in applied research by means of a computer.

2. About the Operational Calculus of Heaviside-Mikusiński.

2.1. Main features. Mikusiński started from the classical Duhamel convolution (see [22])

$$(2.1) \qquad (f * g)(t) = \int_0^t f(t - \tau)g(\tau)d\tau,$$

considering the space $C[0, \infty)$ of the continuous functions on $[0, \infty)$ as a ring on $\mathbb{R}$ or $\mathbb{C}$. Further, he used the fact that due to the Titchmarsh theorem the operation (2.1) has no divisors of zero. In the same way, as the ring $\mathbb{Z}$ of the integers is extended to the field $\mathbb{Q}$ of the rational numbers, Mikusiński extended the ring $(C[0, \infty), *)$ to the smallest field $\mathcal{M}$ containing the initial ring. We denote it by $\mathcal{M}$ and name it Mikusiński's field. The elements of $\mathcal{M}$ are convolution fractions

$$\frac{f}{g} = \frac{\{f(t)\}}{\{g(t)\}},$$

called "operators".

In Mikusiński's calculus each function $f : [0, \infty) \to \mathbb{R}$ is considered as an algebraic object and the notation $f = \{f(x)\}$ is used.

The basic operator in the Mikusiński approach is the integration operator

$$lf(t) = \int_0^t f(\tau)d\tau.$$

In fact, l is the convolution operator $l = \{1\} *$.

The algebraic analogue of the differentiation operator $D = \dfrac{d}{dt}$ is the convolution fraction

$$s = \frac{1}{l},$$

which is not an operator in the proper sense of the word, but an algebraic object.

The relation between the derivative $f'(t)$ and the product $s\{f(t)\}$ is given by the **basic formula of Mikusiński's operational calculus**

(2.2) $$\{f'(t)\} = s\{f(t)\} - f(0),$$

where $f \in C^1[0,\infty)$, and $f(0)$ is considered a "numerical operator".

If the function $f = \{f(t)\}$ has continuous derivatives up to n-th order for $0 \le t < \infty$, a more general formula can be derived:

(2.3) $$f^{(n)} = s^n f - \sum_{i=0}^{n-1} s^i f^{(n-1-i)}(0), \ n = 1, 2, 3, \ldots$$

In the next subsection we consider the use of this formula in the frames of Mikusiński's approach for solving initial value problems for linear ordinary differential equations (LODE) with constant coefficients.

2.2. Solving initial value problems for LODE with constant coefficients using Mikusiński's operational calculus approach. Let $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ be a non-zero polynomial of n-th degree, where the coefficients $a_0, a_1, \ldots, a_n$ are real or complex numbers.

Consider the following Cauchy problem (such problems are known as initial value problems as well):

(2.4) $$P\left(\frac{d}{dt}\right) y = f(t), \ y(0) = \gamma_0, \ y'(0) = \gamma_1, \cdots, \ y^{(n-1)}(0) = \gamma_{n-1}.$$

Using the main formulae (2.2)–(2.3) of the operational calculus of Mikusiński, an "algebraization" of the problem could be made. Then problem (2.4) reduces to the following single algebraic equation of the 1st degree:

(2.5) $$P(s)y = f + Q(s),$$

where

$$P(s) = \sum_{j=1}^{n} a_j s^j, \quad Q(s) = \sum_{j=1}^{n}\left(\sum_{k=j}^{n} a_{n-k}\,\gamma_{k-j}\right)s^{j-1}, \deg Q < \deg P.$$

The formal solution of the above equation has the form

(2.6) $$y = \frac{1}{P(s)}f + \frac{Q(s)}{P(s)}.$$

Further, we can decompose $1/P(s)$ and $Q(s)/P(s)$ into elementary fractions. These fractions can be interpreted as functions using formulae such as ([22]):

$$\frac{1}{(s-\alpha)^n} = \left\{ \frac{t^{n-1}}{(n-1)!}\, e^{\alpha t} \right\}, \quad n = 1, 2, \ldots$$

$$\frac{1}{s^2 + \beta^2} = \left\{ \frac{1}{\beta} \sin \beta t \right\}, \quad \beta > 0$$

(2.7)

$$\frac{s}{s^2 + \beta^2} = \left\{ \cos \beta t \right\}$$

$$\frac{s-\alpha}{(s-\alpha)^2 + \beta^2} = \left\{ e^{\alpha t} \cos \beta t \right\}, \quad etc.$$

Thus we obtain the functions

(2.8)
$$1/P(s) = G(t), \quad Q(s)/P(s) = R(t)$$

and the functional solution has the form

(2.9)
$$y(t) = G(t) * f(t) + R(t).$$

The last step is to compute the convolution product, denoted by $*$ in (2.9).

We just described the main steps of the Heaviside algorithm for solving initial value problems for linear ordinary differential equations with constant coefficients in an interval.

2.3. Initial value problems for systems of ordinary linear differential equations with constant coefficients. Consider a system of n linear ordinary differential equations with constant coefficients:

(2.10)
$$P_{11}\left(\frac{d}{dt}\right) y_1 + P_{12}\left(\frac{d}{dt}\right) y_2 + \ldots + P_{1n}\left(\frac{d}{dt}\right) y_n = f_1(t)$$
$$P_{21}\left(\frac{d}{dt}\right) y_1 + P_{22}\left(\frac{d}{dt}\right) y_2 + \ldots + P_{2n}\left(\frac{d}{dt}\right) y_n = f_2(t)$$
$$\ldots$$
$$P_{n1}\left(\frac{d}{dt}\right) y_1 + P_{n2}\left(\frac{d}{dt}\right) y_2 + \ldots + P_{nn}\left(\frac{d}{dt}\right) y_n = f_n(t)$$

with n unknown functions $y_1, y_2, \ldots y_n$, where P_{ij}, $i, j = 1, 2, \ldots, n$, are polynomials with deg $P_{ij} = n_{ij}$ and $f_i(t)$ are continuous functions for $t \geq 0$.

Let the following initial value conditions be given:

$$y_j^{(i)}(0) = g_{ij}, \; i = 0, 1, \ldots, \max(n_{ij} - 1), \; j = 1, 2, \ldots, n$$

After the "algebraization" of the system (as in the case of one equation), each of the equations assumes the form:

$$\sum_{j=1}^{n} (P_{i,j}(s)y_i(t) + Q_{i,j}(s)) = f_i, \quad i = 1, \ldots, n,$$

where $P_{ij}(s)$ and $Q_{ij}(s)$ are polynomial expressions of s.

In the special case of zero initial conditions ($g_{ij} = 0$ for all i, j), the further considerations are more transparent. In this case we will have $Q_{i,j} = 0$ for $i, j = 1, 2, \ldots, n$.

Denoting by Δ the coefficient's matrix, we can write the following form of the system:

$$(2.11) \qquad\qquad\qquad \Delta(s)\, y = f,$$

where

$$y = \begin{pmatrix} y_1(t) \\ \ldots \\ y_n(t) \end{pmatrix} \qquad f = \begin{pmatrix} f_1(t) \\ \ldots \\ f_n(t) \end{pmatrix}$$

In the field $\mathcal{M}$ this is a system of n linear algebraic equations with n unknowns and we can solve it in the usual way. Thus we can write

$$(2.12) \qquad\qquad\qquad y = \Delta^{-1}(s)\, f,$$

in the case of a non-singular matrix $\Delta(s)$ of the system (2.11).

The solution has the form:

$$(2.13) \qquad\qquad y_k = \sum_{j=1}^{n} \frac{A_{j\,k}(s)}{\det\Delta(s)} * f_j, \; k = 1, 2, \ldots, n,$$

where $A_{jk}(s)$ are the adjoints of the elements $P_{j\,k}$ of matrix Δ. The sign $*$ stands for the Duhamel convolution.

Since we assumed that $\det\Delta(s) \neq 0$, the rational expressions $\dfrac{A_{j\,k}(s)}{\det\Delta(s)}$ in (2.13) can be decomposed into sums of elementary fractions and these fractions can be interpreted as in the case of one equation.

Let us denote $G_{j,k}(t) = \dfrac{A_{jk}(s)}{\det\Delta(s)}$, obtained as the result of the interpretation of the "multipliers" of f_j, $j = 1, 2, \ldots, n$ in the convolution expressions (2.13). Then the solution of the system will obtain the form:

$$y_k(t) = \sum_{j=1}^{n} G_{jk}(t) * f_j(t), \;\; k = 1, 2, \ldots, n,$$

i.e.

$$y_k(t) = \sum_{j=1}^{n} \int_0^t G_{jk}(t-\tau) f_j(\tau)\, d\tau, \;\; k = 1, 2, \ldots, n.$$

We have a Duhamel-type representation of the solution and after computation of all integrals in it we will obtain the final solution of the system (2.10).

To conclude, we can note, that for solving an initial value problem for a system of ordinary linear differential equations with constant coefficients, all steps of the Heaviside algorithm can be performed in a similar way as in case of solving the initial value problem for one equation. This holds not only in the special case of zero initial conditions we considered above, but in the general case as well.

Remarks. The Operational calculus of Mikusiński, presented in [22] is built for continuous functions on the real half–line $[0,\infty)$. Later, in [23], Mikusiński extended his calculus for continuous functions on a finite interval $[0, a]$ and the considered Heaviside algorithm holds as in the case of the infinite interval $[0, \infty)$. In addition, in the same paper it is shown that in the case of the infinite interval this calculus can be replaced by the Laplace transformation, but in the case of a finite interval the Laplace transformation is of no use. This means that the operational calculus of Mikusiński can have wider application than the Laplace transformation.

2.4. Implementation of the Heaviside algorithm.

2.4.1. General remarks. Our implementation of all steps of the Heaviside algorithm is described. Such a complete implementation of the algorithm has not been published yet. The use of the system Macsyma at a step of the algorithm is mentioned in [16].

Having in mind the kind of operations of the Heaviside algorithm and the features of the computer algebra system Mathematica (see [32], [33]), we decided to choose this system for the implementation of the algorithm. Thus an user of the Heaviside algorithm will also be able to use all capabilities of this powerful computing environment.

2.4.2. Steps of the algorithm and their implementation. We will repeat once again the successive steps of the Heaviside algorithm with considerations of their implementation. The case of one equation will be considered in more details.

Step 1. *Algebraization of the problem.* The developed program transforms (2.4) into (2.5). The formula (2.3) and the initial values of (2.4) are applied by means of *Mathematica* rules and functions.

Step 2. *Formation of the polynomials $P(s)$ and $Q(s)$.* They are selected as coefficients to the entries of the unknown function of degree 1 and 0 in (2.5).

Having $P(s)$ and $Q(s)$, we have in fact the formal presentation (2.6) of the solution.

Step 3. *Factorization of the polynomial $P(s)$.* In light of section 2.2. we need $1/P(s)$ and $Q(s)/P(s)$ to be presented as sums of partial fractions. This means that $P(s)$ has to be factorized. *Mathematica* provides a function (named *Factor*) which factors a polynomial over the integers. The syntax of this function allows specifying an appropriate extension field, but in general we don't know this field. That's why we combine the use of the *Mathematica* functions *Solve* and *NSolve* (for solving the polynomial equation $P(\lambda) = 0$) and the function *Factor* as well, thus obtaining a representation of $P(s)$ as a product of factors, each of which is a polynomial of first or second degree, raised to an integer positive power. This process may not finish with success if some of the coefficients of $P(s)$ are parameters and at the same time $\deg P > 4$. In this case the solving of the problem (2.4) is aborted.

Step 4. *Partial fraction decomposition of $\dfrac{1}{P(s)}$ and $\dfrac{Q(s)}{P(s)}$.* It is easy to perform this step if the factorization process on the previous step is finished successfully.

The *Mathematica* function *Apart* represents a rational expression as a sum of terms with minimal denominators. After application of this function to $\dfrac{1}{P(s)}$ and $\dfrac{Q(s)}{P(s)}$ (where $P(s)$ is in factored form), we obtain the partial fraction decomposition of $\dfrac{1}{P(s)}$ and $\dfrac{Q(s)}{P(s)}$. The use of function *Apart* only, without performing Step 3, wouldn't return such representations in some cases.

Step 5. *Interpretation of the result of step 4.* Each fraction in the expressions $\dfrac{1}{P(s)}$ and $\dfrac{Q(s)}{P(s)}$ has to be interpreted as a function using formulae, such as (2.7). We use the main part of the Mikusinski's table (excluding the special functions). The formulae are presented using *Mathematica* rules and ap-

propriate pattern matching. For the cases, when the denominator is a second degree polynomial, it is transformed before the application of the corresponding formulae. If such a polynomial is raised to a (positive) power, convolution powers are computed.

Special attention is paid to this step in order to achieve efficient uniform interpretation of each fraction.

Step 6. *Final form of the solution: $y = G(t) * f(t) + R(t)$.*

In accordance with the introduced notation (2.8), $G(t)$ is the result of the interpretation of $\dfrac{1}{P(s)}$, and $R(t)$ is the result of interpretation of $\dfrac{Q(s)}{P(s)}$. These expressions are already obtained and we have the final representation of the solution but one more computation has to be performed in it: the operation $*$ standing for the Duhamel convolution.

Step 7. *Computation of the convolution product* in the obtained form of the solution. It is defined by a definite integral and the *Mathematica* function *Integrate* is used for its computation. If the form of the result is not "nice" (due to the right-hand side of the equation(s)), numerical integration can be used.

Step 8. *Final result: either the solution or a message that the problem can not be solved.* We mentioned above (in the description of Step 3) when the problem will not be solved in case of one equation. In case of solving the initial value problem for a system of equations, a similar situation may occur. In addition, the problem will not be solved if $det\Delta(s) = 0$ (see (2.11) in subsection 2.3).

2.4.3. An illustrative example. We illustrate the considered steps of finding the solution of an initial value problem by an example. The program implementation of the Heaviside algorithm is used.

We have to solve the following initial value problem:

$$y^{(5)}(t) - 2\,y''(t) - y'(t) + 2\,y(t) \;=\; \frac{e^t}{2}$$

$$\{y(0) \;=\; 1, y'(0) \;=\; 0, y''(0) \;=\; 0, y^{(3)}(0) \;=\; 0, y^{(4)}(0) \;=\; 0\}$$

The result of the algebraization of the left-hand side of the given equation:

$$1 - s^4 + 2\,y - s\,y + s^5\,y - 2\left(-s + s^2\,y\right)$$

Formation of the polynomials P and Q:

$$P \;=\; 2 - s - 2\,s^2 + s^5$$
$$Q \;=\; 1 + 2\,s - s^4$$

Factorization of P:

$$(-1+s)^2\,(1+s)\,(2+s+s^2)$$

Decomposition of $\dfrac{1}{P}$:

$$\frac{1}{8\,(-1+s)^2} - \frac{5}{32\,(-1+s)} + \frac{1}{8\,(1+s)} + \frac{6+s}{32\,(2+s+s^2)}$$

Decomposition of $\dfrac{Q}{P}$:

$$\frac{1}{4\,(-1+s)^2} - \frac{9}{16\,(-1+s)} - \frac{1}{4\,(1+s)} + \frac{-2-3\,s}{16\,(2+s+s^2)}$$

Interpretation of $\dfrac{1}{P}$:

$$G = \frac{1}{8\,e^t} - \frac{5\,e^t}{32} + \frac{e^t\,t}{8}$$

$$+\frac{1}{32}\left(\frac{12\,\sin(\frac{\sqrt{7}\,t}{2})}{\sqrt{7}\,e^{\frac{t}{2}}} - \frac{-7\,\cos(\frac{\sqrt{7}\,t}{2}) + \sqrt{7}\,\sin(\frac{\sqrt{7}\,t}{2})}{7\,e^{\frac{t}{2}}}\right)$$

Interpretation of $\dfrac{Q}{P}$:

$$R = \frac{-1}{4\,e^t} - \frac{9\,e^t}{16} + \frac{e^t\,t}{4}$$

$$+\frac{1}{16}\left(-\frac{4\,\sin(\frac{\sqrt{7}\,t}{2})}{\sqrt{7}\,e^{\frac{t}{2}}} + \frac{3\left(-7\,\cos(\frac{\sqrt{7}\,t}{2}) + \sqrt{7}\,\sin(\frac{\sqrt{7}\,t}{2})\right)}{7\,e^{\frac{t}{2}}}\right)$$

Computation of the convolution product $G(t) * f$:

$$\frac{1}{1792}\left(7\,e^t\,(7+4\,t\,(-5+2\,t)) -\right.$$

$$e^{-\frac{t}{2}}\left(49\,\cos\left(\frac{\sqrt{7}\,t}{2}\right) + 13\,\sqrt{7}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)\right) + 112\,\sinh(t)\Bigg)$$

Final solution:

$$y(t) = \frac{1}{1792}\left(-504e^{-t} - 903\,e^t + 308\,e^t\,t + 56\,e^t\,t^2\right) -$$

$$-\frac{1}{1792}\left(385\,e^{-\frac{t}{2}}\,\cos\left(\frac{\sqrt{7}\,t}{2}\right) - 29\,\sqrt{7}\,e^{\frac{t}{2}}\,\sin\left(\frac{\sqrt{7}\,t}{2}\right)\right)$$

Remarks. In the Heaviside algorithm the initial value conditions are supposed to be given in the point 0. It is easy to develop an extension of the algorithm allowing the initial value conditions to be given in point $t_0 \neq 0$.

2.4.4. Program package for the Heaviside algorithm. The program implementation of the Heaviside algorithm we just described is developed as a *Mathematica* program package. Its main function *DSolveOC* defines a successive performance of all steps of the Heaviside algorithm. The call of this function is similar to the call of the *Mathematica* function *DSolve*. The initial value problem to be solved is given at the input and its solution (or a message that the solution can not be obtained) is returned at the output.

Some additional operations are defined (for visualization of the solution and some others) by means of options.

The following examples illustrate the use of the main function of the package:

Example 1. *Initial value problem for one LODE with constant coefficients*

```
DSolveOC[{y'''''[t] + 2 y'''[t] + y'[t] == α t + β Sin[t] + γ Cos[t],
    y[0] == a₀, y'[0] == a₁, y''[0] == a
```

$$y[t] \to \frac{1}{8} \left(4 \left(-4 + t^2 \right) \alpha + 8\beta + \left(16\alpha + \left(-8 + t^2 \right) \beta - 3 t \gamma \right) \text{Cos}[t] + (3\gamma - t (-4\alpha + 5\beta + t\gamma)) \text{Sin}[t] \right) + a_0 +$$

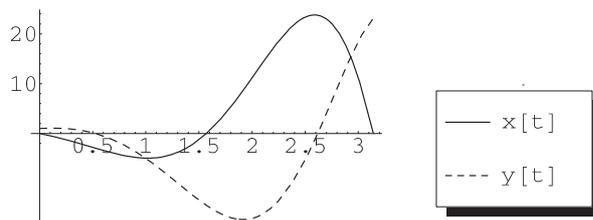$$\text{Sin}[t] \, a_1 + 2 \, a_2 - \frac{1}{2} \left(t \, \text{Cos}[t] - \text{Sin}[t] \right) (a_1 + a_3$$

Example 2. *Initial value problem for a system of LODE with constant coefficients; an option for visualization of the solution is used*

```
mysyst = {-x[t] + 2 y[t] + x'[t] == -2 eᵗ, -2 x[t] - y[t] + y'[t] == 0, x[0] == 0, y[0] == 1};

DSolveOC[mysyst, {x[t], y[t]}, t, GraphInterval → {0, π}]
```

$$\{x[t] \to -4 \, e^t \, \text{Cos}[t] \, \text{Sin}[t], \, y[t] \to e^t \, (-1 + 2 \, \text{Cos}[2 \, t])\}$$

```
Visualization of the solution:
```

2.4.5. Conclusion about the Heaviside algorithm, its implementation and experimental use. Comparing the Heaviside algorithm and its implementation with classical algorithms and those implemented in *Mathematica*, the following notes can be made:

    ▪ the Heaviside algorithm gives a closed–form solution of an initial value problem for a linear ordinary differential equation with constant coefficients or a system of such equations in a direct way, without trying to find a special and the general solution;

    ▪ an uniform approach is used both for homogenous and for non-homogenous equations;

    ▪ no special requirements are posed either to the right-hand side function (as in the use of Laplace transformation) or on the interval where it is defined;

    ▪ during the experimental use of the package many examples were run using the Heaviside algorithm, the *Mathematica* function *DSolve* and a program providing the use of Laplace transformation and inverse Laplace transformation; in many cases we obtained the same result, sometimes advantages of our package were discovered; for some examples *Mathematica* provides a numerical solution only, but we can obtain a closed–form solution; in the case when the right-hand side is not Laplace transformable, we can solve the problem, etc.;

    ▪ in the case of solving an initial–value problem for a system of linear ordinary differential equations, the problem will not be solved if the matrix of the coefficients to the unknown functions is singular; (i.e. $\det \Delta(s) = 0$ in (2.11)); using *Mathematica*, it is easy to compute the rank of this matrix and to express in an appropriate way the solution of the algebraic system and afterwards to interpret it, thus obtaining a "partial" solution of the initial–value problem;

    ▪ it was convenient to implement and test the algorithm in the *Mathematica* environment; its use in this powerful computing environment gives additional possibilities to the users.

3. Extension of the Heaviside algorithm to a class of boundary value problems for LODE with constant coefficients.

3.1. Periodic solutions of LODE with constant coefficients. An auxiliary boundary value problem. An extension of the Heaviside–Mikusiński operational calculus is developed by I. Dimovski and S. Grozdev (see [2, 11]) and in the framework of this operational calculus an extension of the Heaviside algorithm is proposed. It is intended for solving nonlocal initial value problems for LODE with constant coefficients. This approach is used for obtaining periodic solutions of such equations.

Let's consider again a non-zero polynomial with constant coefficients of degree n:
$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$
and the following ordinary linear differential equation with constant coefficients:

(3.14)
$$P\left(\frac{d}{dt}\right)y = f(t), \quad -\infty < t < \infty$$

We are looking for a periodic solution $y(t)$ with period T of this equation, i.e. a solution satisfying the identity:

(3.15)
$$y(t + T) = y(t), \quad -\infty < t < \infty$$

An obvious necessary condition for the existence of a periodic solution of (3.14) with period T is for the function $f(t)$ to be periodic with period T, i.e. for each $t \in \mathbb{R}$:

(3.16)
$$f(t + T) = f(t)$$

The following theorem could be proven: *A solution of* (3.14) *with periodic right-hand side* $f(t)$ *with period* T *is* T-*periodic if and only if the following "boundary" conditions are satisfied:*

(3.17) $$y(T) - y(0) = 0, \ y'(T) - y'(0) = 0, \ldots y^{(n-1)}(T) - y^{(n-1)}(0) = 0$$

This theorem allows the problem of obtaining periodic solutions of (3.14) to be reduced to the problem of finding a solution of this equation in the interval $(-\infty, \infty)$, satisfying the "boundary" conditions (3.17).

Further we reduce this problem to the following intermediate (auxiliary) boundary-value problem:

$$P\left(\frac{d}{dt}\right)y = f(t), \quad -\infty < t < \infty$$
$$\int_0^T y(\tau)\, d\tau = \alpha_0, \ y^{(k)}(T) - y^{(k)}(0) = \alpha_{k+1}, \ k = 0, 1, \ldots n - 2.$$

3.2. Convolution of Dimovski. An operational method for solving the auxiliary problem. The Heaviside algorithm is developed for solving initial value problems for LODE with constant coefficients and it can not be used directly for finding periodic solutions of such equations.

Kaplan (see [20]) uses finite Fourier transform for obtaining periodic solutions (in a rather complicated way in the resonance case), Rosenvasser ([25]) also uses finite Fourier transform, Lurie in [21] uses Laplace transform. We use an alternative direct approach, similar to those of Mikusiński, but using another convolution, based on the operational calculus of Dimovski (see [2]) and related to the nonlocal boundary value problem in $C(\mathbb{R})$:

$$y' = f(x), \int_0^T y(\tau)d\tau = 0,$$

where T is a constant.

The solution

$$L f(t) = \int_0^t f(\tau)d\tau - \frac{1}{T}\int_0^T \left(\int_0^\tau f(\sigma)d\sigma\right) d\tau$$

is an analogue of the integration operator $lf(t) = \int_0^t f(\tau)d\tau$ of Mikusiński's operational calculus.

The operational calculus of Dimovski for the operator L is an analogue of the operational calculus of Mikusiński, but the following convolution of Dimovski [2] is used:

$$(f \overset{t}{*} g)(t) = \Phi_\tau\{\int_\tau^t f(t+\tau-\sigma)g(\sigma)d\sigma\},$$

with an arbitrary linear functional Φ in $C(\mathbb{R})$. In our case the functional $\Phi\{f\} = \frac{1}{T}\int_0^T f(\tau)d\tau$ is used. The convolution

$$(f \overset{t}{*} g)(t) = \frac{1}{T}\int_0^T \left(\int_\tau^t f(t+\tau-\sigma)g(\sigma)d\sigma\right) d\tau$$

has the property $Lf(t) = \{1\}\overset{t}{*} f$.

Dimovski and Grozdev proposed (see [12]) a simpler convolution (without using of repeated integrals):

$$(f * g)(t) = \frac{f(t)}{T}\int_0^T g(\tau)\,d\tau + \frac{g(t)}{T}\int_0^T f(\tau)\,d\tau$$

(3.18)

$$-\frac{1}{T}\int_0^t f(t-\tau)\,g(\tau)\,d\tau - \frac{1}{T}\int_t^T f(t+T-\tau)\,g(\tau)\,d\tau,$$

for which $\{1\} \overset{t}{*} f = f$.

The constant function $\{1\}$ plays the role of a unity in the convolution algebra $(\mathcal{C}(\mathbb{R}), *)$. The operator L has the following representation:

$$L\{1\} = t - \frac{T}{2}, \text{ i.e. } Lf = \left\{ t - \frac{T}{2} \right\} \overset{t}{*} f.$$

Further, convolution fractions of the form f/g are considered (with $f, g \in C[0, T]$, g beeing a nondivisor of 0 of the operation (3.18)). The ring of the continuous functions on $(-\infty, \infty)$ is extended to the smallest ring $\mathcal{M}$, containing the convolution fractions $\dfrac{f}{g}$ with denominators which are nondivisors of 0. The most important convolution fraction

$$S = \frac{1}{L}$$

is considered as an algebraic analogue of d/dt.

The basic formula of the Operational Calculus of Dimovski is:

$$(3.19) \qquad \{f'(t)\} = S\{f(t)\} - \frac{1}{T} \int\limits_0^T f(\tau)d\tau.$$

Here $\dfrac{1}{T} \displaystyle\int\limits_0^T f(\tau)d\tau$ is considered as a constant function.

For $f^{(n)}$ the following formula can be derived from (3.19):

$$(3.20) \quad f^{(n)} = S^n f - \frac{S^n}{T} \int_0^T f(\tau)d\tau - \sum_{k=1}^{n-1} \frac{S^k}{T} \left(f^{(n-1-k)}(T) - f^{(n-1-k)}(0) \right)$$

For the case $T = 1$, the integral operator L is called by Dimovski and Grozdev **Bernoullian integration operator** due to the following relation with the polynomials of Bernoulli (see [2, 13]):

$$L^n\{1\} = \frac{T^n}{n!} B_n \left(\frac{t}{T} \right), \; n = 0, 1, 2, \ldots,$$

where $B_n(t)$ is the polynomial of Bernoulli of degree n.

Further we can follow the scheme of Mikusiński, using the convolution (3.18) (see [28]) and taking into account the following differences:

1) The operation (3.18) has a unit element.

2) This operation has divisors of 0.

We mentioned above that the unity of the convolution algebra $(\mathcal{C}(\mathbb{R}), *)$ is the constant function $\{1\}$.

The eigenfunctions of L are divisors of 0 of (3.18) (see [2], sections 1.3.3 and 2.5). These functions can be found (see [2], section 2.5 and [12]) – they have the form $\varphi_n(t) = Ce^{\frac{2\pi i n t}{T}}$, $n \in \mathbb{Z} \setminus \{0\}$.

For the application of the new operational calculus it is important we to have formulae for convolution fractions of the type $\dfrac{1}{(S - \lambda)^k}$, $k \in \mathbb{N}$. They exist iff $S - \lambda$ is a nondivisor of 0 and this is not true iff $\lambda = \dfrac{2\pi i n}{T}$ and $n \in \mathbb{Z} \setminus \{0\}$ (see [2]).

Thus for each $\lambda \neq \dfrac{2\pi i n}{T}$, $n \in \mathbb{Z} \setminus \{0\}$ the following formulae hold:

$$(3.21) \qquad \frac{1}{S - \lambda} = -\frac{1}{\lambda} + \frac{T e^{t\lambda}}{e^{\lambda T} - 1}$$

$$(3.22) \qquad \frac{S}{S - \lambda} = \frac{T \lambda e^{t\lambda}}{e^{\lambda T} - 1}$$

Corollary. *If $\lambda \neq \dfrac{2\pi i n}{T}$, $n \in \mathbb{Z} \setminus \{0\}$, more general formulae hold (for each integer $k \geq 1$):*

$$(3.23) \qquad \frac{1}{(S - \lambda)^k} = \frac{1}{(k - 1)!} \frac{\partial^{k-1}}{\partial \lambda^{k-1}} \left(-\frac{1}{\lambda} + \frac{T e^{t\lambda}}{e^{\lambda T} - 1} \right)$$

$$(3.24) \qquad \frac{S}{(S - \lambda)^k} = \frac{1}{(k - 1)!} \frac{\partial^{k-1}}{\partial \lambda^{k-1}} \left(\frac{T \lambda e^{t\lambda}}{e^{\lambda T} - 1} \right)$$

The formulae (3.21)–(3.24) are intended to be used for interpretation of rational expressions in the extended Heaviside algorithm. For the purposes of the program implementation of this algorithm additional formulae were derived–for the case when the denominator is an integer power of a second degree polynomial.

3.3. Non-resonance case. Let's apply the Operational Calculus of Dimovski, considered above, for solving the auxiliary problem

(3.25)
$$P\left(\frac{d}{dt}\right) y = f(t), \quad -\infty < t < \infty$$
$$\int_0^T y(\tau)\, d\tau = \alpha_0, \ y^{(k)}(T) - y^{(k)}(0) = \alpha_{k+1}, \ k = 0, 1, \ldots n - 2,.$$

formulated in section 3.1.

Using the formulae (3.19)–(3.20), we can make again an "algebraization" of problem (3.25), thus reducing it to one algebraic equation of 1st degree:

(3.26)
$$P(S)y = f + S\,Q(S),$$

where $P(S)$ and $Q(S)$ are polynomials of S and the degree of $Q(S)$ is less than the degree of $P(S)$.

The formal solution of the above equation has the form

(3.27)
$$y = \frac{1}{P(s)} f + S\frac{Q(s)}{P(s)}.$$

The above representation contains division by $P(S)$ and this is possible if $P(S)$ is not a divisor of 0 in $\mathcal{M}$, i.e. iff $P\left(\frac{2\pi im}{T}\right) \neq 0$ for each $m \in \mathbb{Z} \setminus \{0\}$. This is the so–called **non-resonance case**.

Now we can state **the main steps of the extended Heaviside algorithm** for solving the intermediate problem in the non-resonance case. They are as follows:

1) Finding the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the equation $P(\lambda) = 0$.

2) Finding out that none of the roots have the form $\dfrac{2\pi im}{T}$ with $m \in \mathbb{Z} \setminus \{0\}$.

3) Finding the polynomial $Q(S)$.

4) Expanding $\dfrac{1}{P(S)}$ and $\dfrac{Q(S)}{P(S)}$ into a sum of partial fractions.

5) Interpretation of the fractions $w = \dfrac{1}{P(S)} \quad v = S\dfrac{Q(S)}{P(S)}$ as functions.

6) Representation of the solution in the form $u = w * f + v$.

In comparison with the classical Heaviside algorithm we have here an additional step-this is step 2); on step 5) we have to use other interpretation formulae (such as (3.21)–(3.24)); for computation of the operation $*$ on step (6) we have to use the "new" convolution (3.18).

3.4. Resonance case. If the above condition $\lambda \neq \dfrac{2\pi i n}{T}$, $n \in \mathbb{Z} \setminus \{0\}$ fails for one or more roots of P, we have the so-called **resonance case** and the corresponding roots are called resonance roots.

Let's denote with $n_1, n_2, \ldots, n_p$ all integer numbers, for which $P\left(\dfrac{2\pi i n_k}{T}\right) = 0$, $k = 1, 2, \ldots, p$, and let $C_{n_1, n_2, \ldots, n_p}$ be the subalgebra of $(C(\mathbb{R}), *)$, such that the convolution (3.18) plays the role of multiplication in it.

It was mentioned above that the eigenfunctions of the operator L have the form $\varphi_n(t) = e^{\frac{2\pi i n t}{T}}$, $n \in \mathbb{Z} \setminus \{0\}$. It is shown in [13], that if $f \in C[0, T]$, then

$$f * \{e^{\frac{2\pi i n t}{T}}\} = \chi_n(f)\, e^{\frac{2\pi i n t}{T}}, \ n = \pm 1, \pm 2, \ldots,$$

where

$$(3.28) \qquad \chi_n(f) = \frac{1}{T} \int_0^1 (e^{\frac{2\pi i n t}{T}} - 1) f(t)\, dt, \ n = \pm 1, \pm 2, \ldots,$$

is a complete system of multiplicative functionals. We call them Fourier coefficients of f with respect to $\left\{e^{\frac{2\pi i n t}{T}}\right\}$, $n \in \mathbb{Z} \setminus \{0\}$.

Due to a theorem proven in [13], at least one of the Fourier coefficients of the function f has to be equal to zero in order for this function to be a divisor of 0 in the algebra $(C(\mathbb{R}), *)$. One can prove that this condition is necessary as well.

Let's denote by $\tilde{L}$ the restriction of the operator L to $C_{n_1, n_2, \ldots, n_p}$. Then instead of $Lf = r * f$, for $r(t) = t - \dfrac{T}{2}$ in $[0, T]$, the following presentation in $C_{n_1, n_2, \ldots, n_p}$ will hold: $\tilde{L}f = \tilde{r} * f$, where

$$\tilde{r}(t) = r(t) - \sum_{k=1}^{p} \chi_{n_k}(r) e^{\frac{2\pi i n_k t}{T}} = t - \frac{T}{2} - \sum_{k=1}^{p} \frac{T}{2\pi i n_k t} e^{\frac{2\pi i n_k t}{T}}.$$

We denote by $\mathcal{M}_{n_1, n_2, \ldots, n_p}$ the ring of the convolution fractions of $C_{n_1, n_2, \ldots, n_p}$, whose denominators are nondivisors of 0 of the convolution (3.18). Denote the algebraic inverse element of $\tilde{L}$ by $\tilde{S}$, i.e. $\tilde{S} = \dfrac{1}{\tilde{L}}$.

Two important theorems proven in [13], [12], and [2], are denoted here by T1 and T2, respectively:

Theorem T1. *The elements* $\tilde{S} - \dfrac{2\pi i n_k}{T}, k = 1, 2, \ldots, p$ *of the ring* $\mathcal{M}_{n_1, n_2, \ldots, n_p}$ *are reversible and*

$$(3.29) \qquad \frac{1}{(\tilde{S} - \frac{2\pi i n_k}{T})^m} = \left\{ \frac{(-1)^{m-1}}{\left(\frac{2\pi i n_k}{T}\right)^m} + \frac{e^{\frac{2\pi i n_k t}{T}}}{m!} B_m\left(\frac{t}{T}\right) \right\} *$$

for $m = 1, 2, \ldots$, *where* B_m *is the polynomial of Bernoulli of degree* m *(the sign* $*$ *means a convolution operator).*

Theorem T2. *If* $P\left(\dfrac{2\pi i n_k}{T}\right) = 0$ *for* $k = 1, 2, \ldots, p$ *and* $P\left(\dfrac{2\pi i n}{T}\right) \neq 0$ *for all other integer numbers* $n \neq 0$, *an necessary and sufficient condition for solvability of* (3.25) *is:*

$$(3.30) \qquad \frac{1}{T} \int_0^1 f(t) \left(e^{\frac{2\pi i n_k t}{T}} - 1\right) dt = 0, \; k = 1, 2, \ldots, p,$$

i.e. for the Fourier coefficients of $f(t)$ *with numbers* $n_1, n_2, \ldots, n_p$ *to be equal to* 0.

We can formulate now the algorithm for solving (3.25) in the resonance case:

1) As in the non-resonance case, we can make an algebraization of the problem, i.e. we can reduce it to a single equation but in $C_{n_1, n_2, \ldots, n_p}$:

$$(3.31) \qquad\qquad\qquad P(\tilde{S})\,\tilde{y} = f + Q(\tilde{S}).$$

2) We consider the homogenous BVP:

$$P\left(\frac{d}{dt}\right) y = 0, \; \int_0^T y(\tau)d\tau = 0, \; y^{(j)}(T) - y^{(j)}(0) = 0, j = 0, 1, \ldots n - 2.$$

It is equivalent to the equation $P(\tilde{S})\,y = 0$ and its solutions have the form:

$$y = \left\{ C_1 e^{\frac{2\pi i k_1 t}{T}} + \cdots + C_m e^{\frac{2\pi i k_m t}{T}} \right\},$$

where $C_1, C_2, \ldots, C_m$ are constants.

The solution of (3.25) has the form:

(3.32) $$y = \tilde{y} + \left\{ C_1 e^{\frac{2\pi i k_1 t}{T}} + \cdots + C_m e^{\frac{2\pi i k_m t}{T}} \right\},$$

where $\tilde{y}$ is the solution of (3.31).

3.5. Reducing the problem for obtaining periodic solutions of LODE with constant coefficients to the auxiliary problem.

3.5.1. The case $P(0) \neq 0$. We are looking for a periodic solution of the equation

$$P\left(\frac{d}{dt}\right) y = f(t), \quad -\infty < t < \infty$$

i.e. for a solution satisfying the condition $y(t + T) = y(t)$.

This problem is equivalent to the BVP:

(3.33) $P\left(\dfrac{d}{dt}\right) y = f(t), \ y^{(k)}(T) - y^{(k)}(0) = 0, \ k = 0, 1, 2, \ldots, n-1.$

The problem (3.33) differs from (3.25) by the lack of the boundary condition $\displaystyle\int_0^T y(\tau)\,d\tau = \alpha_0$ and the presence of the additional condition $y^{(n-1)}(T) - y^{(n-1)}(0) = 0$.

If $P(0) \neq 0$, then from (3.33) we can obtain α_0 :

$$\alpha_0 = \frac{1}{P(0)} \int_0^T f(\tau)\,d\tau,$$

i.e. in case $P(0) \neq 0$, the problem for obtaining periodic solution of the equation $P\left(\dfrac{d}{dt}\right) y = f(t)$ with period T is equivalent to the auxiliary problem:

(3.34)

$$P\left(\frac{d}{dt}\right) y = f(t), \quad \int_0^T y(\tau) = \frac{1}{P(0)} \int_0^T f(\tau)\,d\tau$$

$$y(T) - y(0) = 0, \, y'(T) - y'(0) = 0, \, \ldots, \, y^{(n-2)}(T) - y^{(n-2)}(0) = 0$$

3.5.2. The case $P(0) = 0$. Let $P(0) = 0$, i.e. $a_n = 0$.
We consider the more general case when $a_{n-k+1} = \cdots = a_n = 0$ for $k \geq 1$.

Then $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-k}\lambda^k$, and the equation has the form

(3.35) $$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-k} y^{(k)} = f(t)$$

For the existence of periodic solution with period T, it is not sufficient for f to be only periodic with period T. It is necessary the for following condition to be satisfied as well:

(3.36) $$\int_0^T f(\tau)d\tau = 0.$$

Let the above condition be satisfied. We denote with α_0 the unknown number $\int_0^T y(\tau)d\tau$, where $y(t)$ is supposed to be a periodic function with period T.

Consider the auxiliary problem:

$$a_0 \, y^{(n)} + a_1 \, y^{(n-1)} + \cdots + a_{n-k} \, y^{(k)} = f(t)$$
$$y^{(n-2)}(T) - y^{(n-2)}(0) = 0, \ldots, y(T) - y(0) = 0, \quad \int_0^T f(\tau)d\tau = \alpha_0$$

After its algebraization we have

$$P(S)y = f + \frac{\alpha_0}{T}P(S)$$

If $P(S)$ is not a divisor of 0, then the following representation of y holds:

$$y = \frac{1}{P(S)} f + \frac{\alpha_0}{T}.$$

Here $\dfrac{1}{P(S)} = \{G(t)\}$ is a solution of the problem

$$P\left(\frac{d}{dt}\right)G = 1, \quad \int_0^T G(\tau)\, d\tau = 0,$$

$$G(T) - G(0) = 0 \ldots G^{(n-2)}(T) - G^{(n-2)}(0) = 0.$$

Then

$$y = \{G(t)\} * \{f(t)\} + \frac{\alpha_0}{T} =$$

$$(3.37) \qquad\qquad = -\frac{1}{T} \int_0^T f(t-\tau)\, G(\tau)\, d\tau + \frac{\alpha_0}{T},$$

since we supposed that $f(t)$ is periodic with period T and $\int_0^T f(\tau)\, d\tau = 0$.

The function $G(t)$ has the form (see [2]):

$$G(t) = \frac{1}{S^k Q(S)} = \frac{T^k}{k!}\, B_k(\frac{t}{T}) * G_1(t),$$

where B_k stands for the k-th polynomial of Bernoulli and $G_1(t) = \frac{1}{Q(S)}$ is the solution of the following auxiliary problem:

$$a_0\, G_1^{(n-k)} + a_1\, G_1^{(n-k-1)} + \cdots + a_{n-k}\, G_1 = 1, \quad \int_0^T G_1(\tau) = 0,$$

$$G_1(T) - G_1(0) = 0, \ldots, G_1^{(n-k-2)}(T) - G_1^{(n-k-2)}(0) = 0$$

Actually, the condition (3.36) is not only necessary but sufficient as well for the existence of a periodic solution. But this solution is not unique. It is defined by (3.37) up to the arbitrary additive constant $\frac{\alpha_0}{T}$. Thus we have a Duhamel-type representation of the solution of our problem.

3.6. General algorithm for obtaining a periodic solution.

1. Algebraization of the given problem and finding roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the equation $P(\lambda) = 0$

2. a) Finding out roots of the form $\dfrac{2\pi i m}{T}$ $(m \in \mathbb{Z} \setminus \{0\})$.

   b) Verifying whether the roots selected in 2 a) satisfy the conditions (3.30). If for some of the selected roots these conditions are not satisfied, periodic solutions do not exist.

3. Forming the polynomial $Q(S)$.

4. Partial fraction decomposition of $\dfrac{1}{P(S)}$ and $\dfrac{Q(S)}{P(S)}$ and separation of the resonance and non-resonance parts.

5. Interpretation of the fractions $w = \dfrac{1}{P(S)}$ and $v = \dfrac{Q(S)}{P(S)}$ as functions. As was mentioned above, different groups of formulae are used for interpretation of the fractions from the resonance and the non-resonance parts.

6. Presentation of the solution in the form:

$$u_{nr} = w_1 * f + v_1, \quad u_r = w_2 * f + v_2$$
$$u = u_{nr} + u_r,$$

(3.38)

where w_1 and w_2 are functions, obtained at step 5) after interpretation respectively of the non-resonance and resonance parts of the partial fraction decomposition of w; v_1 and v_2 are functions obtained at step 5) after interpretation respectively of the non-resonance and resonance parts of the partial fraction decomposition of v.

The general solution u is the sum of both parts of the solution–the non-resonance part u_{nr} and the resonance part u_r. It is possible, of course, for each of these parts to be equal to zero (the case is determined at steps 1 and 2).

3.7. Program implementation of the algorithm.

3.7.1. General remarks. The program implementation of the general algorithm follows the successive steps formulated above. For obtaining both parts of the solution, the non-resonance and the resonance one, the extended algorithm of Heaviside, described in section 3.3 is used. Its implementation is in fact a modified implementation of the classical algorithm of Heaviside, considered in section 2. The main differences are as follows:

(i) For algebraization of the problem the formula (3.19) is used now.

(ii) Other interpretation formulae are used here. The main formulae mentioned above are (3.21)–(3.24) and (3.29). For practical applications more formulae based on them are derived (see [29]).

(iii) The operation denoted by $*$ in (3.38) is the convolution (3.18). For the application of some of the above formulae convolution powers are computed as in case of the use of Duhamel convolution.

(iv) The verification of conditions (3.30) here is a part of the algorithm.

The implementation of the general algorithm considered above includes finding periodic solutions of systems of linear ordinary differential equations with constant coefficients in a similar way as the implementation of the original Heaviside algorithm includes solving initial value problems for systems of linear ordinary differential equations with constant coefficients.

Main part of the interpretation formulae used by our program implementation:

For the non-resonance case:

$$\frac{1}{(S-a)^m} = \frac{(-1)^m}{a^m} + \frac{T\,\partial_{\{a,-1+m\}}\frac{e^{t\,a}}{-1+e^{T\,a}}}{(-1+m)\,!}$$

$$\frac{1}{S^2+a^2} = \frac{1}{a^2} - \frac{T\,\mathrm{Cos}[a\,t-\frac{a\,T}{2}]\,\mathrm{Csc}[\frac{a\,T}{2}]}{2\,a}$$

$$\frac{S}{S^2+a^2} = \frac{1}{2}\,T\,\mathrm{Csc}\big[\frac{a\,T}{2}\big]\,\mathrm{Sin}\big[a\,t-\frac{a\,T}{2}\big]$$

$$\frac{c\,S+d}{S^2+a^2} = c\left(\frac{1}{a^2} - \frac{T\,\mathrm{Cos}[a\,t-\frac{a\,T}{2}]\,\mathrm{Csc}[\frac{a\,T}{2}]}{2\,a}\right) + d\left(\frac{1}{2}\,T\,\mathrm{Csc}\big[\frac{a\,T}{2}\big]\,\mathrm{Sin}\big[a\,t-\frac{a\,T}{2}\big]\right)$$

$$\frac{1}{S^2+p\,S+q} = \frac{1}{q} + \frac{\left(\frac{e^{\frac{1}{2}\left(-p+\sqrt{p^2-4\,q}\right)t}}{-1+e^{\frac{1}{2}\left(-p+\sqrt{p^2-4\,q}\right)T}} - \frac{e^{-\frac{1}{2}\left(p+\sqrt{p^2-4\,q}\right)t}}{-1+e^{-\frac{1}{2}\left(p+\sqrt{p^2-4\,q}\right)T}}\right)T}{\sqrt{p^2-4\,q}}\,,\quad p^2-4\,q \neq 0$$

For $\sqrt{p^2-4\,q} = \delta$, $p+\delta = \alpha$, $-p+\delta = \beta$:

$$\frac{1}{S^2+p\,S+q} = \frac{1}{q} + \frac{T}{\delta}\,\frac{\left(-\frac{e^{-\frac{t\,\alpha}{2}}}{-1+e^{-\frac{T\,\alpha}{2}}} + \frac{e^{\frac{t\,\beta}{2}}}{-1+e^{\frac{T\,\beta}{2}}}\right)}{\sqrt{p^2-4\,q}}$$

$$\frac{S}{S^2+p\,S+q} = \frac{T}{\delta}\left(\frac{e^{-\frac{t\,\alpha}{2}}\,\alpha}{2\left(-1+e^{-\frac{T\,\alpha}{2}}\right)} + \frac{e^{\frac{t\,\beta}{2}}\,\beta}{2\left(-1+e^{\frac{T\,\beta}{2}}\right)}\right)$$

For the resonance case:

$$\frac{1}{S^m} = \frac{T^m}{m\,!}\,B\big[m,\,\frac{t}{T}\big]$$

$$\frac{1}{(S-a)^m} = \frac{(-1)^m}{a^m} + \frac{e^{a\,t}}{m\,!}\,T^m\,B\big[m,\,\frac{t}{T}\big]$$

$$\frac{1}{S^2+a^2} = \frac{1}{a^2} + \frac{t\,\mathrm{Sin}[a\,t]}{a} - \frac{T\,\mathrm{Sin}[a\,t]}{2\,a}$$

$$\frac{S}{S^2+a^2} = t\,\mathrm{Cos}[a\,t] - \frac{1}{2}\,T\,\mathrm{Cos}[a\,t] + \frac{\mathrm{Sin}[a\,t]}{a}$$

$$\frac{c\,S+d}{S^2+a^2} = c\left(\frac{1}{a^2} + \frac{t\,\mathrm{Sin}[a\,t]}{a} - \frac{T\,\mathrm{Sin}[a\,t]}{2\,a}\right) + d\left(t\,\mathrm{Cos}[a\,t] - \frac{1}{2}\,T\,\mathrm{Cos}[a\,t] + \frac{\mathrm{Sin}[a\,t]}{a}\right)$$

$$\frac{1}{S^2+p\,S+q} = \frac{e^{-\frac{1}{2}t\,(p+\delta)}}{2\,q\,\delta}\left(e^{t\,\delta}\,q\,(2\,t-T) + q\,(-2\,t+T) + 2\,e^{\frac{1}{2}t\,(p+\delta)}\,\delta\right),\quad p^2-4\,q \neq 0$$

3.7.2. Program package. The developed program package for *Mathematica* provides a definition of all described operations of the general algorithm for obtaining periodic solutions of LODE with constant coefficients.

The main function of the package is called *DSolveOCP* and its use is similar to the use of the function *DSolveOC* considered above. An additional argument is the period T. Due to the above considerations, the boundary conditions have the form $y^{(k)}(T) - y^{(k)}(0) = \alpha_{k+1}, \ k = 0, 1, \ldots n-2; \quad \alpha_0$ is computed by the program. The use of an option for visualization of the solution, together with the right-hand side function is provided.

Some illustrative examples follow–for the non-resonance and the resonance cases and for the "mixed" case when the solution is a sum of two parts–resonance and non-resonance ones.

Example for the non-resonance case:

```
<< DSolveOCPpack`
```

Example1 : $\{y(t)\,a^2 + y''(t) == \sin(t), \ \alpha(1) == 0\}; \ T = 2\,\pi$

```
DSolveOCP[{y''[t] + a^2 y[t] == Sin[t], α[1] == 0}, y[t], t, 2 π]
```

$$y[t] \rightarrow \frac{1}{2\,a^3\,(-1+a^2)^2\,\pi}\,(2\,a^2\,\text{Cos}[t]\,\text{Sin}[a\,\pi]\,((-1+a^2)\,\pi\,\text{Cos}[a\,\pi] - 2\,a\,\text{Sin}[a\,\pi]) +$$
$$(a\,(1-3\,a^2+2\,a^4)\,\pi + a\,(-1+a^2)\,\pi\,\text{Cos}[2\,a\,\pi] + (1-3\,a^2)\,\text{Sin}[2\,a\,\pi])\,\text{Sin}[t])$$
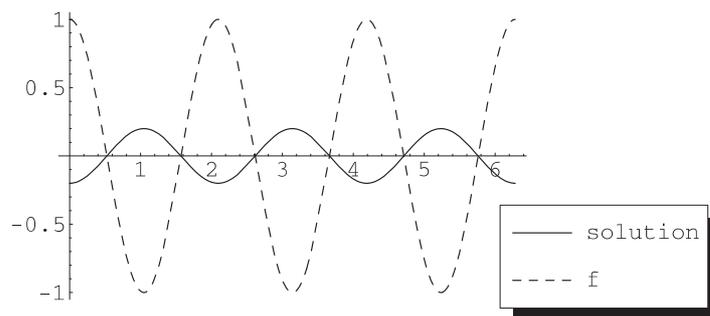
Example for the resonance case (with option for visualization of the solution):

Example2 : $\{4\,y(t) + y''(t) == \cos(3\,t), \ \alpha(1) == 0\}; \ T = 2\,\pi;$

```
DSolveOCP[{y'' (t) + 4 y (t) == Cos[3 t], α[1] == 0},
 y[t], t, 2 π, Graph → True]
```

$$y[t] \rightarrow -\frac{1}{5}\,\text{Cos}[3\,t]$$

```
Visualization of the solution :
```

Example for a "mixed" case:

Example3 : $\{4\,y(t) + 4\,y'(t) + y''(t) + y^{(3)}(t) == \cos(5\,t),\ \alpha(1) == 0,\ \alpha(2) == 0\};\ T = 2\,\pi;$

```
de = y'''[t] + y''[t] + 4 y'[t] + 4 y[t] == Cos[5 t];
```

```
DSolveOCP[{de, α[1] == 0, α[2] == 0}, y[t], t, 2 π]
```

$y[t] \rightarrow \frac{1}{546}\ (\text{-Cos}[5\,t] - 5\,\text{Sin}[5\,t])$

3.8. Advantages of the presented approach for obtaining periodic solutions of LODE with constant coefficients. In the classical methods for finding periodic solutions, at first the general solution is found and after that the periodicity conditions are used for determining the unknown constants in it. In our method the periodicity conditions are taken into account at the level of the algebraization of the problem.

In [11] S. Grozdev compares the method under consideration with the use of Laplace transformation for finding periodic solutions. The main difference is the necessity of the existence of a Laplace transform of the right-hand side of equation. Grozdev discovered advantages of the considered approach in comparison even with the use of the Heaviside–Mikusiński calculus.

We find that the presented approach is more efficient than those in the above mentioned books of Kaplan [20], Rosenvasser [25] and Lurie [21].

The function *DSolve* of *Mathematica* leaves as undetermined the constants appearing in the solution in the resonance case.

4. The Operational calculus approach for solving boundary value problems for some partial differential equations.

4.1. General remarks. If we are interested in the application of the Heaviside-Mikusiński operational calculus to partial differential equations, this calculus should be extended to multivariate functions. Such an extension using the two-dimensional Laplace transformation is proposed in the book [9] by Ditkin and Prudnikov. The principles of the application of multivariate operational calculus for solving Cauchy problems for linear PDE with constant coefficients are developed in Gutterman [17].

We consider here another approach: a way of common use of a combination of two classical methods – the Fourier method [15] and the Duhamel principle [10] – in the frames of a two-dimensional operational calculus suggested by Dimovski [2]. It gives solutions of boundary value problems for some equations of mathematical physics in a closed form.

As is shown below, an extension of the Duhamel principle to the space variables enables one to obtain a closed-form solution of various boundary value problems for some partial differential equations. To this end, as in the classical Duhamel principle, one special solution of the same problem, but for a very simple and special choice of the initial value function, should be obtained using, say, the Fourier method. Then the solution for an arbitrary initial value function can be obtained in the form of a non-classical convolution, using a two-dimensional operational calculus.

Duhamel formulated his principle in 1830 (see [10]). Due to this principle the solution of the boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \ \ u(0,t) = 0, \ u(1,t) = \varphi(t), \ u(x,0) = 0$$

can be obtained for arbitrary $\varphi(t)$, if a solution $U(x,t)$ of the same problem but for a special choice of $\varphi(t)$, namely for $\varphi(t) \equiv 1$, is available. Then the general solution has the form:

$$(4.1) \qquad\qquad u(x,t) = \frac{\partial}{\partial t} \int_0^t U(x,t-\tau)\varphi(\tau)d\tau$$

for $0 \leq x \leq 1, \ 0 \leq t$.

The special solution can be obtained using the Fourier method–it has the form:

$$(4.2) \qquad\qquad U(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 t}\sin n\pi x.$$

Here we are interested in the extension of the Duhamel principle for boundary value problems with non-homogenous initial conditions when the boundary value conditions are homogenous.

4.2. A two-variate operational calculus. Extension of the Duhamel principle. The Duhamel principle can be extended for the space variables for a large class of boundary value problems for linear partial differential equations in finite space domains, in which the Fourier method can be applied. To this end we use an approach, suggested by Dimovski, for extension of the operational calculus of Heaviside–Mikusiński for functions of two variables. This approach can be applied both to local and to non-local boundary value problems (see [5], [6], [7]).

4.2.1. Convolutions for boundary value problems. We consider boundary value problems for three classical equations of mathematical physics in finite domains:

- the heat equation $u_t = u_{xx} + f(x, t)$

- the wave equation $u_{tt} = u_{xx} + f(x, t)$

- the equation of vibrations of a supported beam $u_{tt} = -u_{xxxx} + f(x, t)$.

For solving such problems we need some extensions of the Mikusiński approach using new convolutions. Elements of such operational calculi are presented in [2].

We consider below two types of convolutions, intended for operational calculi for functions of one variable. We will combine them in a convolution for functions of two variables, in order to build operational calculi for functions of two variables.

A. Convolutions for the differentiation operator. The basic BVP for the differentiation operator d/dt in the space $C[0, \infty)$ of the continuous functions $f(t)$, $0 \le t < \infty$ is determined by an arbitrary linear functional χ on $C[0, \infty)$. It looks as follows:

$$(4.3) \qquad\qquad y' = f(t), \ \chi(y) = 0$$

In order for the solution y to exist it is necessary to assume $\chi\{1\} \ne 0$. For simplicity's sake, we take $\chi\{1\} = 1$. Then the solution $y = lf(t)$ could be named a generalized integration operator. Evidently

$$(4.4) \qquad\qquad l\,f(t) = \int_0^t f(\tau)\,d\tau - \chi_\tau\Big\{\int_0^t f(\tau)\,d\tau\Big\}$$

In [2] it is shown that the operation

$$(4.5) \qquad\qquad (f * g)(t) = \chi_\tau\Big\{\int_\tau^t f(t - \sigma + \tau)\,g(\sigma)\,d\sigma\Big\}$$

is a bilinear, commutative and associative operation such that

$$lf = \{1\} * f.$$

In [3] there is developed a one-variate operational calculus based on (4.5) (considered in section 3 of this paper).

B. Convolutions for the square of the differentiation operator.
Let us consider the space $C\,[0, a]$ of the continuous functions on $[0, a]$.

The simplest nonlocal BVP for d^2/dx^2 in $C\,[0, a]$ is given by

$$y'' = f(x), \quad y(0) = 0, \quad \Phi\{y\} = 0$$

where Φ is a linear functional on $C^1\,[0, a]$. In order for it to have a solution, it is necessary to assume $\Phi\{x\} \neq 0$. For simplicity's sake we assume that $\Phi\{x\} = 1$.

Its solution $y = Lf(x)$ has the explicit form

(4.6) $$Lf(x) = \int_0^x (x - \xi)\, f(\xi)\, d\xi - x\, \Phi_\xi\{\int_0^\xi (\xi - \eta)\, f(\eta)\, d\eta\,\}$$

In [2] it is proved that the operation

(4.7) $$(f * g)(x) = -\frac{1}{2}\, \Phi_\xi \left\{ \int_0^\xi h(x, \eta)\, d\eta \right\}$$

where

$$h(x, \eta) = \int_x^\eta f(\eta + x - \zeta)\, g(\zeta)\, d\zeta - \int_{-x}^\eta f(|\eta - x - \zeta|)\, g(|\zeta|)\, \mathrm{sgn}(\eta - x - \zeta)\, \zeta\, d\zeta$$

is a bilinear, commutative and associative operation such that

$$Lf(x) = \{x\} * f$$

4.3. Two-variate convolutions. Operational calculi for l and L in $C = C\,([0, a] \times [0, \infty))$. The idea of a multivariate operational calculus is the following.

Let $u = \{u(x, t)\}$ and $v = \{v(x, t)\}$ be arbitrary functions from the space $C = C\,([0, \infty) \times [0, a])$.

We introduce a bilinear, commutative and associative operation $u * v$ in C such that the operators l and L are multipliers of the convolution algebra $(C, *)$ of the form

$$lu = \{1\} \overset{t}{*} u \quad \text{and} \quad Lu = \{x\} \overset{x}{*} u.$$

Theorem 1. *The operation*

(4.8) $$\{u(x, t)\} * \{v(x, t)\} = -\frac{1}{2}\tilde{\Phi}_\xi \chi_\tau \{h(x, t; \xi, \tau)\}$$

with

$$h(x,t;\xi,\tau) = \int_{\xi}^{x} \int_{\tau}^{t} u(x+\xi-\eta, t+\tau-\sigma)\, v(\eta,\sigma)\, d\sigma d\eta -$$

$$-\int_{-\xi}^{x} \int_{\tau}^{t} u(|x-\xi-\eta|, t+\tau-\sigma)\, v(|\eta|,\sigma)\, \mathrm{sgn}\,[(x-\xi-\eta)\eta]\, d\sigma\, d\eta$$

and with the functional $\tilde{\Phi}_\xi = \Phi \circ \int_0^\xi$ is a convolution of the operators L and l in $C(\Delta)$ (where $\Delta = (0,a] \times [0,\infty)$), for which $L\,l\,u = \{x\} * u$. The operators $lu = \{1\} \overset{(t)}{*} u(x,t)$ and $Lu = \{x\} \overset{(x)}{*} u(x,t)$ are multipliers of this operation.

This theorem gives us an operation $(u * v)(x,t)$ in $C(\Delta)$, which is a convolution of each of the two operators l and L.

Construction of an Operational Calculus for the operators L and l in $C((0,a] \times [0,\infty))$. Consider the ring $\mathfrak{M}$ of the multipliers of the convolution algebra $[C(\Delta), *]$, where $\Delta = [0,a] \times [0,\infty)$.

Denote by $\mathcal{M}$ the ring of the fractions $\dfrac{M}{N}$, where $M \in \mathfrak{M}$, $N \in \mathfrak{M}$, N being non-divisor of 0 in $\mathfrak{M}$. Such fractions are called multipliers fractions.

In $\mathcal{M}$ there can be embedded both the ring $(\mathcal{C}, *)$ and the numerical field ($\mathbb{R}$ or $\mathbb{C}$) and also, the convolution algebras $(\mathcal{C}[0,a], \overset{(x)}{*})$ and $(\mathcal{C}[0,\infty), \overset{(t)}{*})$.

Of course, $\mathfrak{M}$ also is a part of $\mathcal{M}$, since $M = \dfrac{M}{I}$, where I is the identity operator. Hereafter, we will denote I simply by 1.

Let $f = \{f(x)\}$ be a function of the variable x only and $\varphi = \{\varphi(t)\}$–a function of the variable t only, but considered as elements of C.

The operators

$$[f]_t : u \mapsto f \overset{(x)}{*} u$$

and

$$[\varphi]_x : u \mapsto \varphi \overset{(t)}{*} v$$

are said to be numerical operators with respect to t and x respectively. In these notations we have $L = [x]_t$ and $l = [1]_x$. They belongs to $\mathcal{M}$. We denote $s = \dfrac{1}{l}$ and $S = \dfrac{1}{L}$.

The basic formulae of the operational calculus for l and L are

(4.9) $$\frac{\partial u}{\partial t} = su - [\chi_\tau\{u(x,\tau)\}]_t$$

and

$$\frac{\partial^2 u}{\partial x^2} = Su - [\Phi_\xi\{u(\xi,t)\}]_x$$

where the indices t and x mean that the corresponding functions of t and x are considered "partial" numerical operators.

These formulae express the relation between the partial derivatives $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial^2 u}{\partial x^2}$ and the products su and Su, with $s = \dfrac{1}{l}$, $S = \dfrac{1}{L}$.

4.4. Duhamel-type representations of solutions of BVP. In order to illustrate the application of the OC, briefly described above, let's consider the following class of BVP:

$$u_t = u_{xx} + F(x,t),\ 0 < x < a,\ t > 0$$
$$u(0,t) = 0,\ \ \Phi_\xi\{u(\xi,t)\} = 0$$
$$\chi_\tau\{u(x,\tau)\} = f(x),$$

where Φ and χ are linear functionals respectively in $C^1[0,a]$ and $C[0,\infty]$.

Using the main formulae (4.9), we reduce the problem to the single equation:

$$(s - S)u = [f(x)]_t + \{F(x,t)\}$$

Assuming that $s - S$ is not a divisor of 0 (this assumption is equivalent to the requirement for uniqueness of the solution), we can write the following form of the solution in $\mathcal{M}$:

$$(4.10) \qquad u = \frac{1}{s-S}[f(x)]_t + \frac{1}{s-S}\{F(x,t)\}$$

Consider the partial solution $\Omega(x,t)$ of the equation for $F(x,t) \equiv 0$ and $f(x) \equiv x$. This solution is an algebraic object and it has the form:

$$(4.11) \qquad \Omega = \frac{1}{S(s-S)},$$

since $[f(x)]_t = [x]_t = \dfrac{1}{S}$.

Theorem 2. *If $\Omega(x,t)$ is a function in $C(\Delta)$, the problem*

$$u_t = u_{xx},\ \ u(0,t) = 0,\ \ \Phi_\xi\{u(\xi,t)\} = 0,\ \ \chi_\tau\{u(x,\tau)\} = f(x)$$

with $f(0) = 0$, $\Phi\{f\} = 0$ and $f \in C^2[0,a]$ has a classical solution $u(x,t)$ of the form

$$(4.12) \qquad u(x,t) = \frac{\partial^2}{\partial x^2} \left\{ \Omega(x,t) \overset{(x)}{*} f(x) \right\}$$

The proof is given in [4].

Having in mind this theorem, the formulae (4.8), (4.12) and also the forms of Ω, Φ_ξ and χ_τ, we can obtain a representation of the solution of given BVP for the heat equation.

In a similar way we can obtain formulae for the solutions of BVP for the wave equation and for the equation of a supported beam. Such formulae were derived during the work presented here. In most cases sequences of transformations and simplifications were made in order to obtain forms convenient for analysis and for program implementation. The obtained representations are given below.

4.5. Solving BVPs for equations of mathematical physics. We consider local and non-local BVPs for the heat equation, for the wave equation and for the equation of a supported beam. In all problems the partial solutions are denoted by Ω and they are obtained in a form of series, once for every problem (with use of the *Mathematica* system).

4.5.1. Heat equation.

A. Local BVP. Consider the following BVP:

$$u_t = u_{xx}, \quad 0 \leq x \leq 1, \ t \geq 0, \ u(0,t) = 0, \ u(1,t) = 0, \ u(x,0) = f(x)$$

Using Theorem 2 and also (4.8) and (4.12), for $\Phi_\xi\{u(\xi,t)\} = u(1,t)$ we can obtain the following form of the solution:

$$(4.13) \qquad u(x,t) = \int_0^1 [\Omega(1-x-\xi,t) - \Omega(1+x-\xi,t)] f(\xi) d\xi,$$

$$\Omega(x,t) = \sum_{n=1}^{\infty} (-1)^n \exp(-n^2\pi^2 t) \cos n\pi x$$

is a solution of the problem for $f(x) = x$.

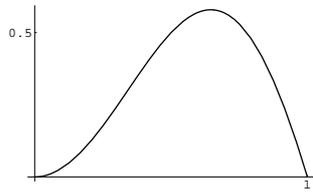An example with $f(x) = x\sin(\pi x)$ (Fig. 1) is considered. The obtained solution has the relief shown on Fig. 2.
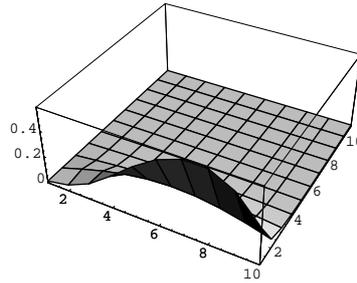
Fig. 1. $f(x)$

Fig. 2. Relief of the solution

B. Non-local BVP. The so-called "Samarskii–Ionkin problem" (see [19]) is considered:

$$u_t = u_{xx},\ u(0,t) = 0,\ \int_0^1 u(x,\tau)d\tau = 0,\ u(x,0) = f(x)$$

We have a BVP with $\Phi_\xi\{u(\xi,t)\} = \int_0^1 f(\xi)d\xi$.

After simplification of (4.12) Dimovski obtains (see [2]):

$$(4.14)\quad u(x,t) = -2\int_0^x \Omega(x - \xi, t)f(\xi)d\xi - \int_x^1 \Omega(1 + x - \xi, t)f(\xi)d\xi$$

$$+ \int_{-x}^1 \Omega(1 - x - \xi, t)f(|\xi|)\mathrm{sgn}\xi d\xi,$$

where

$$\Omega(x,t) = \sum_{n=1}^{\infty} \{-2x\cos 2n\pi x + 8\pi nt \sin 2n\pi x\}\, e^{-4n^2\pi^2 t}$$

This representation of $u(x,t)$ is convenient for numerical computation of an arbitrary number of values of the solution. A visualization of the solution can be made as well.

An example is illustrated on Fig. 4. The boundary function $f(x)$ is "shown" on Fig. 3.

4.5.2. String equation.

A. Local BVP. Consider the BVP:

$$u_{tt} = u_{xx} + F(x,t),\ \ 0 < x < a,\ 0 < t < \infty,$$
$$u(0,t) = 0,\ u(a,t) = 0$$
$$u(x,0) = f(x),\ u_t(x,0) = g(x)$$
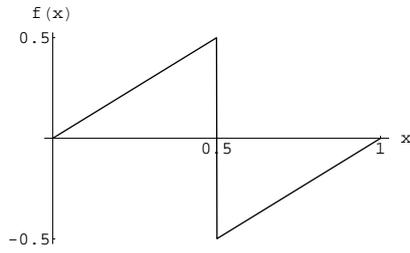
Fig. 3. $f(x)$
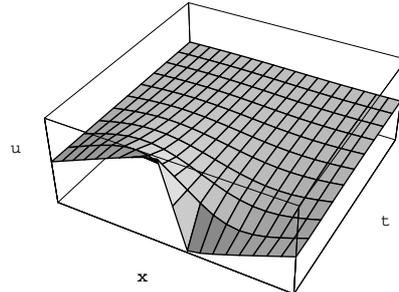
Fig. 4. Relief of the solution

The following representation is obtained for $f(x) \equiv 0$:

$$(4.15) \qquad u(x,t) = -\frac{1}{2}\int_x^1 \Omega(1+x-\xi)g'(\xi)d\xi + \frac{1}{2}\int_{-x}^1 \Omega(1-x-\xi)g'(|\xi|)d\xi,$$

where

$$\Omega(x,t) = \frac{2}{\pi^2}\sum_{n=1}^{\infty}((-1)^{n-1}/n^2)\sin n\pi x \sin n\pi t$$

is a solution of the same problem, but for the special choice $g(x) \equiv x$.

Using this formula, a solution obtained by means of the implemented *Mathematica* system is illustrated in Fig. 6. The function $g(x)$ is visualized in Fig. 5.
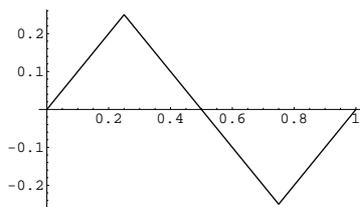
Fig. 5. $g(x)$
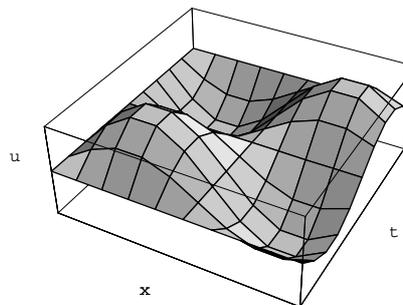
Fig. 6. Relief of the solution

B. Non-local BVPs. Consider the following BVP for the string equation:

$$u_{tt} = u_{xx}, \quad 0 < x < 1, \, 0 < t < \infty$$
$$u(0,t) = 0, \quad \int_0^1 u(\xi,t)d\xi = 0$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

Beilin considers problems of this type (see [1]) and states conditions for the existence and uniqueness of the solution. We derived a formula, convenient for numerical computation of the solution.

Case 1. $f(x) \equiv 0$, $g(x) \neq 0$.

We use the solution $\Omega(x,t)$ for $g(x) \equiv x^3/6 - x/12$ and $f(x) \equiv 0$; we have:

$$\Omega(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{x \, \cos(2\,n\,\pi\,x) \, \sin(2\,n\,\pi\,t)}{4\,n^3\,\pi^3} \right.$$

$$\left. + \left(\frac{t \, \cos(2\,n\,\pi\,t)}{4\,n^3\,\pi^3} - \frac{3\,\sin(2\,n\,\pi\,t)}{8\,n^4\,\pi^4} \right) \sin(2\,n\,\pi\,x) \right\}$$

The solution $u(x,t)$ has the form (see [29])

$$u(x,t) = \frac{\partial^2}{\partial x^2} \left\{ \Omega(x,t) \overset{(x)}{*} g(x) \right\}$$

and after its simplification the concrete form is derived:

$$(4.16) \qquad u(x,t) = -2 \int_0^x \Omega_x(x - \xi, t) g'(\xi) d\xi - \int_x^1 \Omega_x(1 + x - \xi, t) g'(\xi) d\xi +$$

$$+ \int_{-x}^1 \Omega_x(1 - x - \xi, t) g'(|\xi|) d\xi,$$

Using this presentation, a problem was solved for $g(x) = 2\pi x \cos 2\pi x + \frac{3}{2} \sin 2\pi x$. This function and the solution are visualized respectively on Figures 7 and 8.

The numerical solution was compared with the exact solution of the same problem. An error of order 10^{-13} was found. The series for $\Omega(x,t)$ was truncated to $n = 5$.

Case 2. $f(x) \neq 0$ and $g(x) \equiv 0$.

The representation of the solution now has the form (see [29])

$$u(x,t) = \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} \left(\Omega(x,t) \overset{(x)}{*} f(x) \right)$$
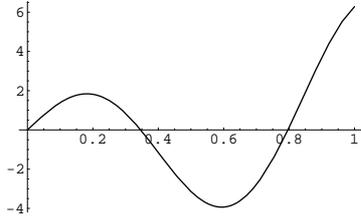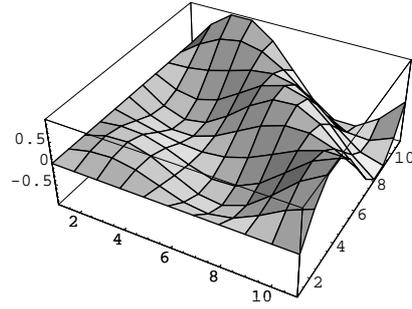
Fig. 7. $g(x)$

Fig. 8. Relief of the solution

For the purposes of simplification of this representation we introduce

$$\tilde{\Omega}(x,t) = \int_0^t \Omega_x(x,\tau)\, d\tau$$

where $\Omega(x,t)$ is a solution of the problem under consideration for the special choice $f(x) \equiv x^3/6 - x/12$ and $g(x) \equiv 0$.

$$\tilde{\Omega}(x,t) = \sum_{n=1}^{\infty} \left\{ -\frac{x \cos(2\,n\,\pi\,x)\sin(n\,\pi\,t)^2}{n^2\,\pi^2} - \right.$$

$$\left. \frac{t\,\cos(n\,\pi\,t)\,\sin(n\,\pi\,t)\,\sin(2\,n\,\pi\,x)}{n^2\,\pi^2} + \frac{\sin(n\,\pi\,t)^2\,\sin(2\,n\,\pi\,x)}{n^3\,\pi^3} \right\}$$

The following representation of $u(x,t)$ is derived:

$$(4.17) \quad u(x,t) = -2\int_0^x \tilde{\Omega}_x(x-\xi,t)\,f''(\xi)\,d\xi - \int_x^1 \tilde{\Omega}_x(1+x-\xi,t)\,f''(\xi)\,d\xi$$

$$+ \int_{-x}^1 \operatorname{sgn} x\,\tilde{\Omega}_x(1-x-\xi,t)\,f''(\xi)\,d\xi - 2\,\tilde{\Omega}(1,t)\,f''(x) + f(x)$$

A visualization of the numerical solution is presented on Fig. 10. On Figure 9 the graph of the employed function $f(x)$ is shown. The series for Ω_x was truncated to $n = 10$.

By comparison of the numerical and exact solutions of the problem an error of order 10^{-9} was found.
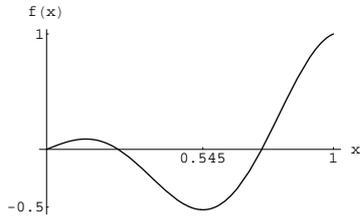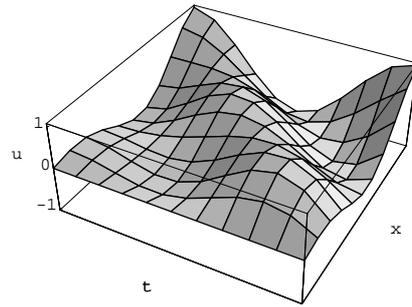
Fig. 9. $f(x)$

Fig. 10. Relief of the solution

4.5.3. Equation of a free supported beam.

A. Local BVPs. Consider the following problem for the equation of a free supported beam (see [14]):

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^4 u}{\partial x^4},\ 0 < x < 1,\ 0 < t < \infty,$$

$$u(0,t) = 0,\ u_{xx}(0,t) = 0,\ u(1,t) = 0,\ u_{xx}(1,t) = 0$$

$$u(x,0) = f(x),\ u_t(x,0) = g(x).$$

For the case $f(x) \equiv 0$ we obtain:

(4.18)

$$u(x,t) = -\frac{1}{2}\int_x^1 \Omega_x(1 + x - \xi, t)\, g(\xi)\, d\xi + \frac{1}{2}\int_{-x}^1 \Omega_x(1 - x - \xi, t) g(|\xi|)\,\mathrm{sgn}\,\xi\, d\xi,$$

where

$$\Omega_x(x,t) = \frac{2}{\pi^2}\sum_{n=1}^{\infty}((-1)^{n-1}/n^2)\sin(n\pi)^2 t\cos n\pi x$$

Numerical values of the solution for the function, visualized in Fig. 11, are computed using this formula. The relief of the solution is illustrated in Fig. 12.

B. Non-local BVPs. Consider the problem

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^4 u}{\partial x^4},\ 0 < x < 1,\ 0 < t < \infty,$$

$$u(0,t) = 0,\ u_{xx}(0,t) = 0$$

$$\int_0^1 u(\xi,t)d\xi = 0,\ u_x(1,t) - u_x(0,t) = 0$$
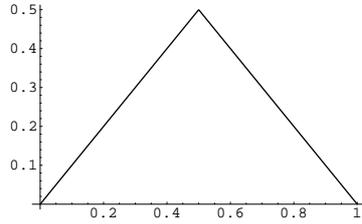
$$u(x,0) = f(x),\ u_t(x,0) = g(x)$$
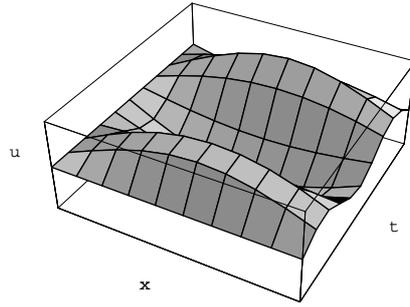
Fig. 11. $f(x)$

Fig. 12. Relief of the solution

Case 1. $f(x) \equiv 0$, $g(x) \neq 0$. The solution $u(x, t)$ in this case has the form (see [29])

$$u(x,t) = \frac{\partial^2}{\partial x^2}\left\{\Omega(x,t) \overset{(x)}{*} f(x)\right\}$$

After some simplifications an explicit representation is obtained:

$$(4.19) \quad u(x,t) = -2\int_0^x \Omega_x(x-\xi,t)g'(\xi)d\xi - \int_x^1 \Omega_x(1+x-\xi,t)g'(\xi)d\xi +$$

$$\int_{-x}^1 \Omega_x(1-x-\xi,t)g'(|\xi|)d\xi$$

where

$$\Omega_x(x,t) = \left\{\sum_{n=1}^{\infty} \frac{\cos(2\,n\,\pi\,x)\,\left(8\,n^2\,\pi^2\,t\,\cos(4\,n^2\,\pi^2\,t) - 3\,\sin(4\,n^2\,\pi^2\,t)\right)}{8\,n^4\,\pi^4} -\right.$$

$$\left.\frac{x\,\sin(4\,n^2\,\pi^2\,t)\,\sin(2\,n\,\pi\,x)}{4\,n^3\,\pi^3}\right)\right\}$$

An experimental computation of the solution using the above formulae was made. By comparison with the exact solution an error of order 10^{-8} was found, for $n = 10$ in the series for $\Omega_x(x,t)$.

A visualization of the solution is illustrated in Fig. 14. The graph of the employed function $g(x)$ is shown in Figure 13.

Case 2. $f(x) \neq 0$, $g(x) \equiv 0$ The following representation is obtained after simplification made in a way, similar to those in *Case 2* for the string equation
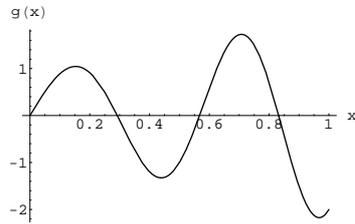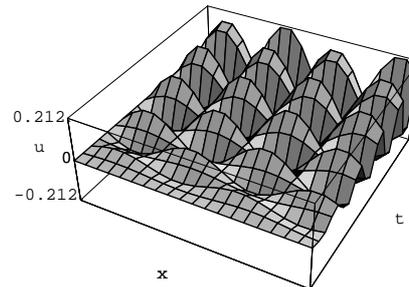
Fig. 13. $g(x)$

Fig. 14. Relief of the solution

(for more details see [29]):

$$(4.20) \quad u(x,t) = -2\int_0^x \tilde{\Omega}_{xx}(x-\xi,t)f^{iv}(\xi)d\xi + \int_x^1 \tilde{\Omega}_{xx}(1+x-\xi,t)f^{iv}(\xi)d\xi -$$

$$\int_{-x}^1 \tilde{\Omega}_{xx}(1-x-\xi,t)f^{iv}(|\xi|)\text{sign}(\xi)d\xi + 2f^{iv}(\xi)(\tilde{\Omega}_x(0,t) - \tilde{\Omega}_x(1,t)) + f(x),$$

where

$$\tilde{\Omega}_{xx}(x,t) = \left\{ -\frac{x\,\cos(2\,n\,\pi\,x)\,\sin(2\,n^2\,\pi^2\,t)^2}{4\,n^4\,\pi^4} \right.$$

$$-\frac{\left(4\,n^2\,\pi^2\,t\,\cos(2\,n^2\,\pi^2\,t) - 3\,\sin(2\,n^2\,\pi^2\,t)\right)\,\sin(2\,n^2\,\pi^2\,t)\,\sin(2\,\pi\,x)}{4\,n^7\,\pi^5}$$

$$\left. -\frac{\sin(2\,n^2\,\pi^2\,t)^2\,\sin(2\,n\,\pi\,x)}{4\,n^5\,\pi^5} \right\}$$

An example for this case is given in the next subsection as an illustration of the use of the developed program package for solving this problem.

For all derived formulae presented in section 4, a comparison with the exact solutions of the BVPs with the solutions computed by means of our formulae, is made. High precision and "good" time of the computations were ascertained.

Let's note that at the moment the system *Mathematica* can't solve nonlocal BVPs.

4.6. Program packages for the considered BVPs. The presentations of the solutions given above are used in 3 developed program packages (for each of the considered equations). Each of the packages has functions for solving local and nonlocal problems.

An example of the use of the package for solving a nonlocal BVP for the equation of a supported beam for $f(x) \neq 0$, $g(x) \equiv 0$ is given. A table of numerical values of the solution in intervals, given by the user, is produced (a part of the table is included in the picture). A visualization of the solution is made together with the boundary function f. The series for the special solution Ω is truncated to $n = 3$. A comparison with the exact solution is made and the minimal and maximal errors are obtained; this comparison is required in the call to the package function named *DSolveOCBeamN* by the last (optional) argument *uexact*.

Illustrative example: solving a nonlocal BVP for the equation of a supported beam:

```
f1[x_] := -2 Sin[4 π x]; uexact = -2 Cos[16 π² t] Sin[4 π x];

<< DSolveOCBeam'

DSolveOCBeamN[{f1, 0}, u, x, t, {0.0000001, 1, 0.1}, {0, 1, 0.1}, 3, uexact]
```

Numerical values of the solution:

-2.51327×10^{-6}	-1.90211	-1.17557	1.17557	1.90211	-2.51327×10^{-6}
2.50454×10^{-6}	1.8955	1.17148	-1.17149	-1.8955	2.50454×10^{-6}
-2.47839×10^{-6}	-1.87571	-1.15925	1.15926	1.87571	-2.47839×10^{-6}
2.43501×10^{-6}	1.84288	1.13896	-1.13897	-1.84288	2.43501×10^{-6}
-2.3747×10^{-6}	-1.79724	-1.11075	1.11076	1.79724	-2.3747×10^{-6}
2.29789×10^{-6}	1.7391	1.07482	-1.07483	-1.7391	2.29789×10^{-6}
-2.2051×10^{-6}	-1.66888	-1.03142	1.03142	1.66887	-2.2051×10^{-6}
2.09697×10^{-6}	1.58705	0.980847	-0.98085	-1.58705	2.09697×10^{-6}
-1.97427×10^{-6}	-1.49418	-0.923454	0.923458	1.49418	-1.97427×10^{-6}
1.83785×10^{-6}	1.39093	0.859642	-0.859645	-1.39093	1.83785×10^{-6}
-1.68865×10^{-6}	-1.27801	-0.789854	0.789856	1.27801	-1.68865×10^{-6}

Visualization of function f and the solution:
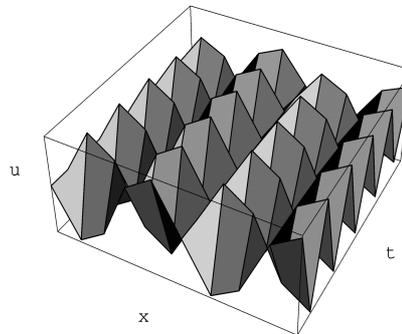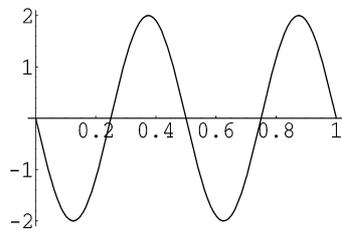
{MinError, MaxError}:

$\{-1.98508 \times 10^{-13}, 1.98394 \times 10^{-9}\}$

5. Concluding remarks. The presented results allow some conclusions to be made.

$\bullet$ The considered operational methods are efficient and convenient for obtaining closed-form solutions and numerical solutions of some mixed boundary value problems.

$\bullet$ Modification of algorithms and development of steps of algorithms based on the considered methods were performed as steps of their program implementation.

$\bullet$ Interpretation formulae providing a complete implementation of the Heaviside algorithm and the modified Heaviside algorithm were derived.

$\bullet$ Duhamel-type representations of the solutions of BVPs, mainly for the string and beam equations, were derived.

$\bullet$ 5 program packages for the computer algebra system *Mathematica*, aimed for solving Cauchy problems and boundary value problems by the operational approach, were developed.

$\bullet$ An experimental proof of the efficiency and the advantages of the considered operational methods was obtained.

$\bullet$ The choice of the computer algebra system *Mathematica* providing efficient symbolic and numerical computations, as well as convenient program language, was an important precondition for the good results of the work.

The use of the implemented packages in the powerful computing environment of *Mathematica* is convenient and efficient. It can be part of a complete problem solving process in research or engineering, using a large scale of *Mathematica* tools.

Further research and applications of the presented results are under way. They are connected with solving other boundary value problems (not considered yet in our work), considering real phenomena (an example can be found in [8]), taking into account some ideas presented in [27], etc.

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REFERENCES

[1] BEILIN S. Existence of solutions for one-dimensional wave equation with nonlocal conditions. *Electronic Journal of Differential Equations*, 2001, No 76, 1–8.

[2] DIMOVSKI, I. H. Convolutional Calculus. Kluwer Acad. Publishers, Dordrecht, 1990.

[3] DIMOVSKI, I. H. Non-local operational calculi. *Proc. Steklov Inst. Math.,* **3** (1995), 53–65.

[4] DIMOVSKI I. Duhamel-Type Representation of the Solutions of Non-Local Boundary Value Problems. In: Proc. of the Second Conference Differential Equations and Applications, Rousse'81, Bulgaria, 1982, 240–247.

[5] DIMOVSKI I., M. SPIRIDONOVA. Computational Approach to Nonlocal Boundary Value Problems by Multivariate Operational Calculus . *Math. Sci. Res. J.*, **9** (2005), No 12, 315–329.

[6] DIMOVSKI I., M. SPIRIDONOVA. Computer Implementation of Solutions of BVP for Finite Vibrating Systems. *Mathematica Balkanica*, **18** (2004), Fasc. 3–4, 277–285.

[7] DIMOVSKI I., M. SPIRIDONOVA. The Fourier Method and the problem of its computer implementation. *Mathematics and Education in Mathematics*, **32** (2003), 117–127.

[8] DIMOVSKI I., M. SPIRIDONOVA. A Feasible Linear Model for Takoma Narrows Bridge Destruction. MASSEE International Congress of Mathematics, Ohrid, Sept. 16–20, 2009, 29–30.

[9] DITKIN V. A., A. P. PRUDNIKOV. Operational Calculus of two variables and its applications. GI Fiz.–mat. lit., Moscow, 1958 (in Russian).

[10] DUHAMEL J. M.-C. Memoire sur le Métode generale relative au mouvement de la chaleur dans les corps solides plongés dans les mileaux dont la temperature varie avec le temps. *Journal de l'Ec. Polyt.,* **14** (1830), 20–77.

[11] GROZDEV S. Convolutional approach to abstract differential equations in the "special" case. PhD Thesis, Sofia, 1980 (in Bulgarian).

[12] GROZDEV S., I. DIMOVSKI. Direct operational method for finding periodic solutions of linear differential equations with constant coefficients. *Mathematics and Education in Mathematics*, **8** (1979), 187–196 (in Bulgarian).

[13] GROZDEV S., I. DIMOVSKI. Bernoullian operational calculus. *Mathematics and Education in Mathematics* **9** (1980), 30–36 (in Bulgarian).

[14] FARLOW S. Partial differential equations for scientists and engineers. John Wiley and Sons, Inc., 1982.

[15] FOURIER J. The analytical theory of heat. Dover, N.Y., 1955.

[16] GLINOS N., B. D. SAUNDERS. Operational Calculus Techniques for Solving Differential Equations. In: , Proceedings EUROSAM 1984 (Ed. J. Fitch), Lcture Notes in Computer Science, **174** (1984), Springer-Verlag, 23–34.

[17] GUTTERMAN M. An operational method in partial differential equations. *SIAM J. Appl. Math.*, **17** (1969), No 2, 468–493.

[18] HEAVISIDE O. Electromagnetic theory, London, 1899.

[19] IONKIN N. I. Numerical solution of nonclassical boundary value problems for the heat equation. *Differential equations*, **13** (1977), No 2, 294–304 (in Russian).

[20] KAPLAN W. Operational Methods for Linear Systems. Addison-Wesley Series in Mathematics, USA, 1962.

[21] LURIE A. I. Operational calculus and its application to problems of mechanics, Publ. house of techn.-theor. lit., Moscow, Leningrad, 1950 (in Russian).

[22] MIKUSIŃSKI J. Operational Calculus, Oxford-Warszawa, 1959.

[23] MIKUSIŃSKI J. Le calcul operatoinnel d'interval fini. *Studia Math.*, **15** (1956), 225–251.

[24] ROSENKRANZ M. A New Symbolic Method for Solving Linear Two-Point BVPs on the Operator Level. *Journal of Symbolic Computation*, **39** (2005), 171–199 (in Russian).

[25] ROZENVASSER E. N., S. K. VOLOVODOV. Operator methods and vibration processes, "Nauka", Moscow, 1985 (in Russian).

[26] SKORNIK K. Professor Jan Mikusinski – Life and work. *Scientiae Mathematicae Japonicae*, **66** (2007), No 1, 1–16.

[27] SPIRIDONOVA M. Applied Computer Algebra Microenvironments. *Mathematica Balkanica*, New Series, **19** (2005), Fasc. 3–4, 445–452.

[28] SPIRIDONOVA M. Computation of Periodic Solutions of Differential Equations by the Operational Calculus Approach. *Mathematics and Education in Mathematics*, **34** (2005), 191–195.

[29] SPIRIDONOVA M. Direct Operational Methods in the Environment of a Computer Algebra System. PhD Thesis, IMI – BAS, Sofia, 2008 (in Bulgarian).

[30] STEPHENS E. The Elementary Theory of Operational Mathematics. McGraw Hill Book Co., Inc., New York and London, 1937.

[31] VASHCHENKO–ZAHARCHENKO M. E. Symbolic computation and its application for integration of differential equations. Ms. Thesis, Kiev, 1862 (in Russian).

[32] Wolfram Research. `http://www.wolfram.com`

[33] WOLFRAM S. Mathematica. A System for Doing Mathematics by Computer. Second Edition, Addison-Wesley Publishing Co., Redwood City, California, 1991.

[34] YOSIDA K., S. OKAMOTO. A Note on Mikusinski's operational calculus. *Proc. Japan Acad. Ser. A Math. Sci.*, **56** (1980), No 1, 1–3.

[35] Yosida K. *Operational Calculus. A Theory of Hyperfunctions.* Springer Verlag, New York Inc., 1984.

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: `mspirid@math.bas.bg`