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ON SOME MODIFICATIONS OF THE NEKRASSOV METHOD FOR NUMERICAL SOLUTION OF LINEAR SYSTEMS OF EQUATIONS*

Anton Iliev, Nikolay Kyurkchiev, Milko Petkov

Abstract. A modification of the Nekrassov method for finding a solution of a linear system of algebraic equations is given and a numerical example is shown.

1. Introduction.

Let us consider the linear system $Ax - b = 0$ or

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ii}x_i + \cdots + a_{in}x_n - b_i = 0 = f_i(x_1, x_2, \ldots, x_n),\quad i = 1, 2, \ldots, n. \tag{1}$$

Suppose that the matrix A is diagonally dominant and $a_{ii} > 0$, $i = 1, \ldots, n$.

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One of the more effective iteration methods for solving the system (1) is the Jacobi procedure (his method is also known as the *method of simultaneous displacements*):

$$\begin{aligned} x_i^{k+1} &= -\sum_{j\neq i}^{n} \frac{a_{ij}}{a_{ii}} x_j^k + \frac{b_i}{a_{ii}} \\ &= x_i^k - \frac{1}{a_{ii}} f_i(x_1^k, \ldots, x_n^k) \\ &= x_i^k - \frac{f_i(x_1^k, \ldots, x_n^k)}{\partial f_i / \partial x_i^k}, \end{aligned} \tag{2}$$

$$i = 1, 2, \ldots, n; \quad k = 0, 1, 2, \ldots,$$

i.e., (2) is *the Newton scheme applied for the equation* $f_i = 0$.

A more powerful class of methods can be described by the recursion (*Richardson iteration*):

$$x^{k+1} = x^k - \alpha_k (A x^k - b), \tag{3}$$

where $\alpha_i,\ i = 1, \ldots, k$ are damping factors.

For instance, the Richardson iteration (3) with the application of *Chebyshev acceleration factors* is defined by

$$\alpha_i = 2\left(a + b - (b - a)\cos\frac{(2i+1)\pi}{2(k+1)}\right)^{-1},$$

$$i = 0, 1, \ldots, k$$

$a \le \lambda_i \le b,\ \ i = 1, \ldots, n$ (λ_i are the eigenvalues of matrix A).

In [8] we give the following modification of the Richardson method:

$$x_i^{k+1} = x_i^k - \frac{1}{M_i^k}\left(\sum_{j=1}^{n} a_{ij} x_j^k - b_i\right), \tag{4}$$

$$i = 1, 2, \ldots, n; \quad k = 0, 1, 2, \ldots,$$

where

$$M_i^k = \prod_{j\neq i}^{n} |x_i^k - x_j^k|, \ \ i = 1, 2, \ldots, n; \ \ k = 0, 1, \ldots.$$

For other contributions see Saad and van der Vorst [14], Freund, Golub and Nachtigal [6], Ishihara, Muroya and Yamamoto [7], Maleev [10], Stork [17], Zawilski [18].

One geometric interpretation of method (4) is also given in [8].

In a similar manner other iterations can be obtained which are modifications of algorithms which have been explored in details in books by Björck [2], Fadeev, D. and Fadeev, V. [4] and Barrett, R., M. Berry and others [1].

As an example a scheme of the Gauss–Seidel or the Nekrassov method (see Nekrassov [13], Mehmke [11] and Nekrassov and Mehmke [12]) look thus:

$$x_i^{k+1} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{k+1} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^k + \frac{b_i}{a_{ii}}, \tag{5}$$

$$i = 1, 2, \dots, n; \;\; k = 0, 1, 2, \dots .$$

2. Main results. Let us explore the following modification of the Nekrassov method (assume that $x_i \neq x_j$ and $x_i^0 \neq x_j^0$ for $i \neq j$):

$$x_i^{k+1} = x_i^k - \frac{1}{N_i^k} \left(\sum_{j=1}^{i-1} a_{ij} x_j^{k+1} + a_{ii} x_i^k + \sum_{j=i+1}^{n} a_{ij} x_j^k - b_i \right), \tag{6}$$

$$i = 1, 2, \dots, n; \quad k = 0, 1, 2, \dots,$$

where

$$N_i^k = \prod_{j=1}^{i-1} |x_i^k - x_j^{k+1}| \prod_{j=i+1}^{n} |x_i^k - x_j^k|, \;\; i = 1, 2, \dots, n; \;\; k = 0, 1, \dots .$$

Let

$$\delta_i^k = \frac{a_{ii}}{N_i^k}, \; i = 1, 2, \dots, n; \; k = 0, 1, 2, \dots .$$

The iteration procedure (6) (*successive overrelaxation procedure*) can be rewritten as

$$\begin{aligned} x_i^{k+1} &= x_i^k - \frac{a_{ii}}{N_i^k} \left(\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{k+1} + x_i^k + \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^k - \frac{b_i}{a_{ii}} \right) \\ &= x_i^k (1 - \delta_i^k) - \delta_i^k \left(\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{k+1} + \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^k - \frac{b_i}{a_{ii}} \right). \end{aligned} \tag{7}$$

1. When $\delta_i^k = 1$ from (7) we obtain the Nekrassov method.

2. One geometric interpretation of method (7) is the following:
Let

$$F_{k,i} = (x - x_1^{k+1}) \dots (x - x_{i-1}^{k+1})(x - x_{i+1}^k) \dots (x - x_n^k).$$

Then

$$F'_{k,i}(x_i^k) = \prod_{j=1}^{i-1} (x_i^k - x_j^{k+1}) \prod_{j=i+1}^{n} (x_i^k - x_j^k)$$

and the previous expression can be used for approximation of a_{ii} in the Nekrassov procedure.

We give a convergence theorem for the relaxation method (7).

Theorem 1. *Let*

(8)
$$\beta_i = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{a_{ii}}, \ \gamma_i = \sum_{j=i+1}^{n} \frac{|a_{ij}|}{a_{ii}}, \ \delta_i^k \in (1, 2),$$
$$\beta_i + \gamma_i \in \left(0, \frac{1 - |1 - \delta_i^k|}{\delta_i^k}\right) \subset (0, 1), \ \ i = 1, 2, \dots, n; \ k = 0, 1, 2, \dots.$$

Then the iteration procedure (7) *converges to the unique solution* x_i, $i = 1, 2, \dots, n$ *of the system* (1).

P r o o f. For the error $x_i^{k+1} - x_i$, we have

(9)
$$\begin{aligned} x_i^{k+1} - x_i &= x_i^k (1 - \delta_i^k) - x_i \\ &\quad - \delta_i^k \left(\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{k+1} + \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^k - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j - x_i \right) \\ &= (x_i - x_i^k)(\delta_i^k - 1) + \delta_i^k \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} (x_j - x_j^{k+1}) + \delta_i^k \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} (x_j - x_j^k) \end{aligned}$$

and

(10)
$$\begin{aligned} |x_i^{k+1} - x_i| &\leq |\delta_i^k - 1||x_i^k - x_i| + \delta_i^k \sum_{j=1}^{i-1} \frac{|a_{ij}|}{a_{ii}} |x_j - x_j^{k+1}| + \delta_i^k \sum_{j=i+1}^{n} \frac{|a_{ij}|}{a_{ii}} |x_j - x_j^k| \\ &\leq |\delta_i^k - 1| ||x - x^k||_1 + \delta_i^k \beta_i ||x - x^{k+1}||_1 + \delta_i^k \gamma_i ||x - x^k||_1 \\ &= \left(|\delta_i^k - 1| + \gamma_i \delta_i^k\right) ||x - x^k||_1 + \delta_i^k \beta_i ||x - x^{k+1}||_1. \end{aligned}$$

Let

$$\max_i |x_i^{k+1} - x_i| = |x_{i_0}^{k+1} - x_{i_0}|.$$

Then from (10) we get

$$\begin{aligned} ||x - x^{k+1}||_1 &= \max_i |x_i - x_i^{k+1}| = |x_{i_0}^{k+1} - x_{i_0}| \\ &\leq \left(|\delta_{i_0}^k - 1| + \gamma_{i_0}\delta_{i_0}^k\right) ||x - x^k||_1 + \delta_{i_0}^k \beta_{i_0} ||x - x^{k+1}||_1 \end{aligned}$$

and

(11) $$||x - x^{k+1}||_1 \leq \frac{|\delta_{i_0}^k - 1| + \gamma_{i_0}\delta_{i_0}^k}{1 - \delta_{i_0}^k \beta_{i_0}} ||x - x^k||_1 = K_{i_0} ||x - x^k||_1.$$

Evidently from (8) we have

$$K_{i_0} = \frac{|\delta_{i_0}^k - 1| + \gamma_{i_0}\delta_{i_0}^k}{1 - \delta_{i_0}^k \beta_{i_0}} \leq \frac{|\delta_{i_0}^k - 1| + \delta_{i_0}^k \left(\frac{1 - |\delta_{i_0}^k - 1|}{\delta_{i_0}^k} - \beta_{i_0}\right)}{1 - \delta_{i_0}^k \beta_{i_0}} = 1.$$

This proves Theorem 1. $\square$

Let

$$L = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{pmatrix}, \ R = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \ X^k = \begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_n^k \end{pmatrix},$$

$$P = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}, \ \delta^k = \begin{pmatrix} \delta_{11}^k & 0 & \cdots & 0 \\ 0 & \delta_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_{nn}^k \end{pmatrix}.$$

Theorem 2. *The iteration* (6) (*or* (7)) *is convergent when all roots (eigenvalues) of the equation*

(12) $$\left|A\delta^k - \left(P + \delta^k L\right) + t\left(P + \delta^k L\right)\right| = 0$$

are $|t_i| < 1, \; i = 1, \ldots, n$.

P r o o f. In matrix terms the *successive overrelaxation procedure* (7) can be written as follows:

$$X^{k+1} = \left(P + \delta^k L\right)^{-1} \left((I - \delta^k)P - \delta^k R\right) X^k + \left(P + \delta^k L\right)^{-1} \delta^k b, \tag{13}$$

i.e.

$$X^{k+1} = BX^k + c.$$

Evidently, $|B - tI| = 0$ can be represented as

$$|B - tI| = \left|\left(P + \delta^k L\right)^{-1}\right| \left|A\delta^k - \left(P + \delta^k L\right) + t\left(P + \delta^k L\right)\right| = 0,$$

and the statement of Theorem 2 follows from the standard iteration theory. □

3. In a number of cases the success of the procedures of type (5) depends on the proper ordering of the equations (and $x_i, \; i = 1, \ldots, n$) in system (1).

In spite of this fact the following variant of the Nekrassov method is known [4]:

$$x_i^{k+1} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^k - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{k+1} + \frac{b_i}{a_{ii}}. \tag{14}$$

Further, we are interested in the *successive overrelaxation procedure* (14) based on the method (7):

$$x_i^{k+1} = x_i^k(1 - \delta_i^k) - \delta_i^k \left(\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^k + \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{k+1} - \frac{b_i}{a_{ii}}\right). \tag{15}$$

In matrix terms the *successive overrelaxation procedure* (15) can be written as follows:

$$X^{k+1} = \left(P + \delta^k R\right)^{-1} \left((I - \delta^k)P - \delta^k L\right) X^k + \left(P + \delta^k R\right)^{-1} \delta^k b. \tag{16}$$

The pseudocode for the modification of Nekrassov method (6) is given in Figure 1.

```
        Choose an initial guess x^0 for the solution x.
for k = 1, 2, ...,
            for  i = 1, 2, ..., n
                 x_i = a_ii x_i^{k-1}
                 N_i^{k-1} = 1
                 for j = 1, 2, ..., i - 1
                     N_i^{k-1} = N_i^{k-1} |x_i^{k-1} - x_j^k|
                     x_i = x_i + a_ij x_j^k
                 end
                 for j = i + 1, ..., n
                     N_i^{k-1} = N_i^{k-1} |x_i^{k-1} - x_j^{k-1}|
                     x_i = x_i + a_ij x_j^{k-1}
                 end
                 x_i = (x_i - b_i)/N_i^{k-1}
            end
        x^k = x^{k-1} - x
        check convergence; continue if necessary
end
```

Fig. 1. The modification of the Nekrassov method (6)

3. Numerical example. As an example we will consider the system:

$$\left|\begin{aligned} x_1 + 3x_2 - 2x_3 &= 5 \\ 3x_1 + 5x_2 + 6x_3 &= 7 \\ 2x_1 + 4x_2 + 3x_3 &= 8 \end{aligned}\right.$$

The exact solution of the system is $x(-15, 8, 2)$.

For an initial approximation we choose $x^0(-15.02, 8.02, 2.02)$.

We give the results of numerical experiments (8 iterations) for each of methods (5) and (6).

In Table 1 the following notations are used:

– in the first column the serial number of the iteration is given;

– using the modified scheme (6) in the second column the obtained results are given (array $x[\,]$);

– using the classical Nekrassov scheme (5) in the third column the obtained results are given (array $y[\,]$).

Table 1

1	$X[1] = -15.02000000000000$ $X[2] = 8.01884259259259$ $X[3] = 2.01906701123844$	$Y[1] = -15.02000000000000$ $Y[2] = 7.98800000000000$ $Y[3] = 2.02933333333333$
2	$X[1] = -15.01999590828629$ $X[2] = 8.01776735852021$ $X[3] = 2.01820333133504$	$Y[1] = -14.90533333333333$ $Y[2] = 7.90800000000000$ $Y[3] = 2.05955555555556$
3	$X[1] = -15.01998800937772$ $X[2] = 8.01676825375272$ $X[3] = 2.01740388676229$	$Y[1] = -14.60488888888888$ $Y[2] = 7.69146666666666$ $Y[3] = 2.14797037037037$
4	$X[1] = -15.01997656688334$ $X[2] = 8.01583967575863$ $X[3] = 2.01666397500912$	$Y[1] = -13.77845925925925$ $Y[2] = 7.08951111111110$ $Y[3] = 2.39962469135803$
5	$X[1] = -15.01996182501415$ $X[2] = 8.01497643146312$ $X[3] = 2.01597923762914$	$Y[1] = -11.46928395061725$ $Y[2] = 5.40202074074072$ $Y[3] = 3.11016164609054$
6	$X[1] = -15.01994400998709$ $X[2] = 8.01417370750044$ $X[3] = 2.01534563522253$	$Y[1] = -4.98573893004107$ $Y[2] = 0.65924938271599$ $Y[3] = 5.11149344307273$
7	$X[1] = -15.01992333132901$ $X[2] = 8.01342704260065$ $X[3] = 2.01475942421623$	$Y[1] = 13.24523873799748$ $Y[2] = -12.68093537448576$ $Y[3] = 10.74442134064936$
8	$X[1] = -15.01989998308720$ $X[2] = 8.01273230196133$ $X[3] = 2.01421713531614$	$Y[1] = 64.53164880475601$ $Y[2] = -50.21229489163284$ $Y[3] = 26.59529398567311$

4. A wide area of problems and practical tasks in tomography and image processing are reduced to the problem of solving a system of algebraic equations with constraint conditions for the initial approximations x_i^0, $i = 1, \ldots, n$ (see Björck [2], A. van der Sluis and H. van der Vorst [16], A. Louis and F. Natterer [9] and R. Santos and A. de Pierro [15]).

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REFERENCES

[1] Barrett R., M. Berry, T. Chan, J. Demmel, J. Donato, J. Dongarra, V. Eijkhout, R. Pozo, C. Romine, H. Van der Vorst. Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods. SIAM, Philadelphia, 1994.

[2] Björck A. Numerical methods for least squares problems. SIAM, Philadelphia, PA, 1996.

[3] De Boor C., J. Rice. Extremal polynomials with application to Richardson iteration for indefinite linear system. Technical Rept., Wisconsin Univ. Madison Mathematics Research Center, 1980.

[4] Fadeev D., V. Fadeev. Numerical methods of linear algebra. Fizmatgiz M., 1963.

[5] Fisher B., L. Reichel. A stable Richardson iteration method for complex linear systems. *Numer. Math.*, **54** (1989), No 2, 225–242.

[6] Freund R., G. Golub, N. Nachtigal. Iterative solution of linear systems. *Acta Numerica*, (1992), 1–44.

[7] Ishihara K., Y. Muroya, T. Yamamoto. On linear SOR-like methods, II-convergence on the SOR-Newton method for mildly nonlinear equations. *Japan J. Indust. Appl. Math.*, **14** (1997), No 1, 99–110.

[8] Kyurkchiev N., M. Petkov, A. Iliev. On a modification of Richardson method for numerical solution of linear system of equations. *Compt. rend. Acad. bulg. Sci.*, **61** (2008), No 10, 1257–1264.

[9] Louis A., F. Natterer. Mathematical problems of computerized tomography. Proceedings of the IEEE, **71** (1983), No 3, 379–389.

[10] Maleev A. A lower bound for order of convergence of methods of Jacobi's type. *Comp. Math. and Math. Phys.*, **46** (2006), 2128–2137 (in Russian).

[11] Mehmke R. On the Seidel scheme for iterative solution of linear system of equations with a very large number of unknowns by successive approximations. *Math. Sb.*, **16** (1892), No 2, 342–345 (in Russian).

[12] Mehmke R., P. Nekrassov, Solution of linear system of equations by means of successive approximations, *Math. Sb.*, **16** (1892), 437–459 (in Russian).

[13] Nekrassov P. Determination of the unknowns by the least squares when the number of unknowns is considerable. *Math. Sb.*, **12** (1885), 189–204 (in Russian).

[14] Saad Y., H. van der Vorst. Iterative solution of linear systems in the 20th century. *JCAM*, **123** (2000), No 1–2, 1–33.

[15] Santos R., A. de Pierro. The effect of the nonlinearity on GCV applied to conjugate gradients in computerized tomography. *Comp. and Appl. Math.*, **25** (2006), No 1, 111–128.

[16] van der Sluis A., H. van der Vorst. Numerical solution of large, sparse linear algebraic systems arising from tomographic problems. In: Seismic Tomography (Ed. G. Notel), D. Reidel Publ. Comp., Dordrecht, The Netherlands (1987), 49–84.

[17] Stork Ch. Comparison of Richardson's iteration with Chebyshev acceleration factors to conjugate gradient iteration. Stanford Expl. Proj. SEP-57 (1988), 479–503.

[18] Zawilski A., Numerical stability of the cyclic Richardson iteration, *Numer. Math.*, **60** (1991), No 1, 251–290.

Anton Iliev
Faculty of Mathematics and Informatics
Paisii Hilendarski University of Plovdiv
24 Tsar Assen Str.
4000 Plovdiv, Bulgaria
e-mail: aii@uni-plovdiv.bg
and
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria

Nikolay Kyurkchiev
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: nkyurk@math.bas.bg

Milko Petkov
Faculty of Mathematics and Informatics
Episkop Konstantin Preslavski University of Shumen
115 Universitetska Str.
9712 Shumen, Bulgaria

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