ON SOME MODIFICATIONS OF THE NEKRASSOV METHOD FOR NUMERICAL SOLUTION OF LINEAR SYSTEMS OF EQUATIONS

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Abstract. A modification of the Nekrassov method for finding a solution of a linear system of algebraic equations is given and a numerical example is shown.

1. Introduction. Let us consider the linear system $Ax - b = 0$ or

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ii}x_i + \cdots + a_{in}x_n - b_i = 0 = f_i(x_1, x_2, \ldots, x_n),$$

$$i = 1, 2, \ldots, n.$$  

Suppose that the matrix $A$ is diagonally dominant and $a_{ii} > 0$, $i = 1, \ldots, n$. 


Key words: Solving linear systems of equations, Jacobi method, Richardson method, Nekrassov method, Chebyshev’s acceleration factors, pseudocode.

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One of the more effective iteration methods for solving the system (1) is the Jacobi procedure (his method is also known as the method of simultaneous displacements):

\[
x^{k+1}_i = -\sum_{j \neq i}^n \frac{a_{ij}}{a_{ii}} x^k_j + \frac{b_i}{a_{ii}}
\]

\[
= x^k_i - \frac{1}{a_{ii}} f_i(x^k_1, \ldots, x^k_n)
\]

\[
= x^k_i - \frac{f_i(x^k_1, \ldots, x^k_n)}{\partial f_i/\partial x^k_i},
\]

\[i = 1, 2, \ldots, n; \quad k = 0, 1, 2, \ldots,
\]

i.e., (2) is the Newton scheme applied for the equation \(f_i = 0\).

A more powerful class of methods can be described by the recursion (Richardson iteration):

\[
x^{k+1} = x^k - \alpha_k (Ax^k - b),
\]

where \(\alpha_i, \ i = 1, \ldots, k\) are damping factors.

For instance, the Richardson iteration (3) with the application of Chebyshev acceleration factors is defined by

\[
\alpha_i = 2 \left( a + b - (b - a) \cos \left( \frac{(2i + 1)\pi}{2(k + 1)} \right) \right)^{-1},
\]

\[i = 0, 1, \ldots, k\]

\(a \leq \lambda_i \leq b, \ i = 1, \ldots, n\) (\(\lambda_i\) are the eigenvalues of matrix \(A\)).

In [8] we give the following modification of the Richardson method:

\[
x^{k+1}_i = x^k_i - \frac{1}{M^k_i} \left( \sum_{j=1}^n a_{ij} x^k_j - b_i \right),
\]

\[i = 1, 2, \ldots, n; \quad k = 0, 1, 2, \ldots,
\]

where

\[M^k_i = \prod_{j \neq i} |x^k_i - x^k_j|, \quad i = 1, 2, \ldots, n; \quad k = 0, 1, \ldots.
\]
For other contributions see Saad and van der Vorst [14], Freund, Golub and Nachtigal [6], Ishihara, Muroya and Yamamoto [7], Maleev [10], Stork [17], Zawilski [18].

One geometric interpretation of method (4) is also given in [8].

In a similar manner other iterations can be obtained which are modifications of algorithms which have been explored in details in books by Björck [2], Fadeev, D. and Fadeev, V. [4] and Barrett, R., M. Berry and others [1].

As an example a scheme of the Gauss–Seidel or the Nekrassov method (see Nekrassov [13], Mehmke [11] and Nekrassov and Mehkte [12]) look thus:

\[ x_{i}^{k+1} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_{j}^{k+1} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_{j}^{k} + \frac{b_{i}}{a_{ii}}, \]

\[ i = 1, 2, \ldots, n; \quad k = 0, 1, 2, \ldots. \]

2. Main results. Let us explore the following modification of the Nekrassov method (assume that \( x_i \neq x_j \) and \( x_i^0 \neq x_j^0 \) for \( i \neq j \)):

\[ x_{i}^{k+1} = x_{i}^{k} - \frac{1}{N_{i}^{k}} \left( \sum_{j=1}^{i-1} a_{ij} x_{j}^{k+1} + a_{ii} x_{i}^{k} + \sum_{j=i+1}^{n} a_{ij} x_{j}^{k} - b_{i} \right), \]

\[ i = 1, 2, \ldots, n; \quad k = 0, 1, 2, \ldots, \]

where

\[ N_{i}^{k} = \prod_{j=1}^{i-1} |x_{i}^{k} - x_{j}^{k+1}| \prod_{j=i+1}^{n} |x_{i}^{k} - x_{j}^{k}|, \quad i = 1, 2, \ldots, n; \quad k = 0, 1, 2, \ldots. \]

Let

\[ \delta_{i}^{k} = \frac{a_{ii}}{N_{i}^{k}}, \quad i = 1, 2, \ldots, n; \quad k = 0, 1, 2, \ldots. \]

The iteration procedure (6) (successive overrelaxation procedure) can be rewritten as

\[ x_{i}^{k+1} = x_{i}^{k} - \frac{a_{ii}}{N_{i}^{k}} \left( \sum_{j=1}^{i-1} \frac{a_{ij} x_{j}^{k+1}}{a_{ii}} + x_{i}^{k} + \sum_{j=i+1}^{n} \frac{a_{ij} x_{j}^{k}}{a_{ii}} - b_{i} \right) \]

\[ = x_{i}^{k} (1 - \delta_{i}^{k}) - \delta_{i}^{k} \left( \sum_{j=1}^{i-1} \frac{a_{ij} x_{j}^{k+1}}{a_{ii}} + \sum_{j=i+1}^{n} \frac{a_{ij} x_{j}^{k}}{a_{ii}} - b_{i} \right). \]
1. When $\delta^k_i = 1$ from (7) we obtain the Nekrassov method.
2. One geometric interpretation of method (7) is the following:

Let

$$F_{k,i} = (x - x_1^{k+1}) \ldots (x - x_{i-1}^{k+1})(x - x_{i+1}^{k+1}) \ldots (x - x_n^k).$$

Then

$$F'_{k,i}(x^k) = \prod_{j=1}^{i-1} (x_i^k - x_j^{k+1}) \prod_{j=i+1}^{n} (x_i^k - x_j^k)$$

and the previous expression can be used for approximation of $a_{ii}$ in the Nekrassov procedure.

We give a convergence theorem for the relaxation method (7).

**Theorem 1.** Let

$$\beta_i = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{a_{ii}}, \quad \gamma_i = \sum_{j=i+1}^{n} \frac{|a_{ij}|}{a_{ii}}, \quad \delta^k_i \in (1, 2),$$

(8)

$$\beta_i + \gamma_i \in \left(0, \frac{1 - |1 - \delta^k_i|}{\delta^k_i}\right) \subset (0, 1), \quad i = 1, 2, \ldots, n; \ k = 0, 1, 2, \ldots.$$

Then the iteration procedure (7) converges to the unique solution $x_i$, $i = 1, 2, \ldots, n$ of the system (1).

**Proof.** For the error $x_i^{k+1} - x_i$, we have

$$x_i^{k+1} - x_i = x_i^{k}(1 - \delta^k_i) - x_i$$

$$= \delta^k_i \left( \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{k+1} + \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{k} - \sum_{j=1}^{n} \frac{a_{ij}}{a_{ii}} x_j - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j - x_i \right)$$

$$= (x_i - x_i^k)(\delta^k_i - 1) + \delta^k_i \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} (x_j - x_j^{k+1}) + \delta^k_i \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} (x_j - x_j^k)$$

(9)

and

$$|x_i^{k+1} - x_i| \leq |\delta^k_i - 1||x_i - x_i^k| + \delta^k_i \sum_{j=1}^{i-1} \frac{|a_{ij}|}{a_{ii}} |x_j - x_j^{k+1}| + \delta^k_i \sum_{j=i+1}^{n} \frac{|a_{ij}|}{a_{ii}} |x_j - x_j^k|$$

$$\leq |\delta^k_i - 1||x - x^k||_1 + \delta^k_i \beta_i ||x - x^{k+1}||_1 + \delta^k_i \gamma_i ||x - x^k||_1$$

$$= (|\delta^k_i - 1| + \gamma_i \delta^k_i) ||x - x^k||_1 + \delta^k_i \beta_i ||x - x^{k+1}||_1.$$
Let
\[ \max_{i} |x_i^{k+1} - x_i| = |x_{i_0}^{k+1} - x_{i_0}|. \]

Then from (10) we get
\[
||x - x^{k+1}||_1 = \max_{i} |x_i - x_i^{k+1}| = |x_{i_0}^{k+1} - x_{i_0}|
\leq (|\delta_{i_0}^k - 1| + \gamma_{i_0} \delta_{i_0}^k) ||x - x^k||_1 + \delta_{i_0}^k \beta_{i_0} ||x - x^{k+1}||_1
\]

and
\[
||x - x^{k+1}||_1 \leq |\delta_{i_0}^k - 1| + \gamma_{i_0} \delta_{i_0}^k \left( \frac{1 - |\delta_{i_0}^k - 1|}{1 - \delta_{i_0}^k \beta_{i_0}} \right) ||x - x^k||_1 = K_{i_0} ||x - x^k||_1.
\]

Evidently from (8) we have
\[
K_{i_0} = \frac{|\delta_{i_0}^k - 1| + \gamma_{i_0} \delta_{i_0}^k}{1 - \delta_{i_0}^k \beta_{i_0}} \leq \frac{|\delta_{i_0}^k - 1| + \delta_{i_0}^k \left( \frac{1 - |\delta_{i_0}^k - 1|}{\delta_{i_0}^k} - \beta_{i_0} \right)}{1 - \delta_{i_0}^k \beta_{i_0}} = 1.
\]

This proves Theorem 1. □

Let
\[
L = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
a_{21} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & 0
\end{pmatrix}, \quad R = \begin{pmatrix}
0 & a_{12} & \cdots & a_{1n} \\
0 & 0 & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}, \quad X^k = \begin{pmatrix}
x_1^k \\
x_2^k \\
\vdots \\
x_n^k
\end{pmatrix},
\]

\[
P = \begin{pmatrix}
a_{11} & 0 & \cdots & 0 \\
0 & a_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{pmatrix}, \quad \delta^k = \begin{pmatrix}
\delta_{11}^k & 0 & \cdots & 0 \\
0 & \delta_{22}^k & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \delta_{nn}^k
\end{pmatrix}.
\]

**Theorem 2.** The iteration (6) (or (7)) is convergent when all roots (eigenvalues) of the equation
\[
|A\delta^k - (P + \delta^k L) + t (P + \delta^k L)| = 0
\]
are \( |t_i| < 1, \ i = 1, \ldots, n \).

**Proof.** In matrix terms the *successive overrelaxation procedure* (7) can be written as follows:

\[
X^{k+1} = (P + \delta^k L)^{-1} \left( (I - \delta^k)P - \delta^k R \right) X^k + (P + \delta^k L)^{-1} \delta^kB,
\]

i.e.

\[
X^{k+1} = BX^k + c.
\]

Evidently, \( |B - tI| = 0 \) can be represented as

\[
|B - tI| = \left| (P + \delta^k L)^{-1} \right| \left| A\delta^k - (P + \delta^k L) + t \left( P + \delta^k L \right) \right| = 0,
\]

and the statement of Theorem 2 follows from the standard iteration theory. \( \square \)

**3.** In a number of cases the success of the procedures of type (5) depends on the proper ordering of the equations (and \( x_i, \ i = 1, \ldots, n \)) in system (1).

In spite of this fact the following variant of the Nekrassov method is known [4]:

\[
x^{k+1}_i = \frac{1}{a_{ii}} \left( \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i+1}^{n} a_{ij}x_j^{k+1} + b_i \right).
\]

Further, we are interested in the *successive overrelaxation procedure* (14) based on the method (7):

\[
x^{k+1}_i = x_i^k (1 - \delta^k_i) - \delta^k_i \left( \sum_{j=1}^{i-1} a_{ij}x_j^k + \sum_{j=i+1}^{n} a_{ij}x_j^{k+1} - b_i \right).
\]

In matrix terms the *successive overrelaxation procedure* (15) can be written as follows:

\[
X^{k+1} = (P + \delta^k R)^{-1} \left( (I - \delta^k)P - \delta^k L \right) X^k + (P + \delta^k R)^{-1} \delta^kB.
\]

The pseudocode for the modification of Nekrassov method (6) is given in Figure 1.
Choose an initial guess $x^0$ for the solution $x$.

for $k = 1, 2, \ldots,$

$$x_i = a_{ii}x_i^{k-1}$$

$N_i^{k-1} = 1$

for $j = 1, 2, \ldots, i - 1$

$$N_i^{k-1} = N_i^{k-1}|x_i^{k-1} - x_j^{k-1}|$$

$$x_i = x_i + a_{ij}x_j^{k-1}$$

end

for $j = i + 1, \ldots, n$

$$N_i^{k-1} = N_i^{k-1}|x_i^{k-1} - x_j^{k-1}|$$

$$x_i = x_i + a_{ij}x_j^{k-1}$$

end

$$x_i = (x_i - b_i)/N_i^{k-1}$$

end

$$x^k = x^{k-1} - x$$

check convergence; continue if necessary

end

Fig. 1. The modification of the Nekrassov method (6)

3. Numerical example. As an example we will consider the system:

$$
\begin{align*}
    x_1 + 3x_2 - 2x_3 &= 5 \\
    3x_1 + 5x_2 + 6x_3 &= 7 \\
    2x_1 + 4x_2 + 3x_3 &= 8
\end{align*}
$$

The exact solution of the system is $x(-15, 8, 2)$.

For an initial approximation we choose $x^0 (-15.02, 8.02, 2.02)$.

We give the results of numerical experiments (8 iterations) for each of methods (5) and (6).

In Table 1 the following notations are used:

– in the first column the serial number of the iteration is given;

– using the modified scheme (6) in the second column the obtained results are given (array $x[]$);

– using the classical Nekrassov scheme (5) in the third column the obtained results are given (array $y[]$).
4. A wide area of problems and practical tasks in tomography and image processing are reduced to the problem of solving a system of algebraic equations with constraint conditions for the initial approximations $x_0^i$, $i = 1, \ldots, n$ (see Björck [2], A. van der Sluis and H. van der Vorst [16], A. Louis and F. Natterer [9] and R. Santos and A. de Pierro [15]).

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Table 1

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REFERENCES


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