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NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXTENDABILITY OF TERNARY LINEAR CODES

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Abstract. We give the necessary and sufficient conditions for the extendability of ternary linear codes of dimension $k \geq 5$ with minimum distance $d \equiv 1$ or $2 \pmod 3$ from a geometrical point of view.

1. Introduction. Let $V(n,q)$ denote the vector space of n-tuples over $\mathrm{GF}(q)$, the finite field of order q. A linear code C is an $[n,k,d]_q$ code over $\mathrm{GF}(q)$ of length n with dimension k whose minimum Hamming distance is d. The *weight* of a vector $\boldsymbol{x} \in V(n,q)$, denoted by $wt(\boldsymbol{x})$, is the number of nonzero coordinate positions in $\boldsymbol{x}$. Let A_i be the number of codewords of C with weight i. We only consider *non-degenerate* codes having no coordinate which is identically zero.

The code obtained by deleting the same coordinate from each codeword of C is called a *punctured code* of C. If there exists an $[n+1,k,d+1]_q$ code C' which gives C as a punctured code, C is called *extendable* (to C') and C' is an *extension* of C. It is well known that a binary linear code with odd d is extendable by adding an overall parity check. The extendability of linear codes has been studied by Hill

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[1, 2], van Eupen and Lisonek [13], Simonis [12] and Maruta [5, 6, 7, 8]. Recently, Kohnert [3] investigates how to get an $[n+l, k, d+s]_q$ code from a non-extendable $[n, k, d]_q$ code ((l, s)-extension).

Let C be an $[n, k, d]_3$ code with $k \geq 3$, $\gcd(3, d) = 1$. We define three non-negative integers Φ_0, Φ_1, Φ_e as follows:

$$\Phi_0 = \frac{1}{2} \sum_{3|i, i \neq 0} A_i, \quad \Phi_1 = \frac{1}{2} \sum_{i \not\equiv 0, d \pmod 3} A_i, \quad \Phi_e = \frac{1}{2} \sum_{d < i \equiv d \pmod 3} A_i,$$

where the notation $x|y$ means that x is a divisor of y. The pair of integers (Φ_0, Φ_1) is called the *diversity* of C. Let $\mathcal{D}_k$ be the set of all possible diversities of such codes. $\mathcal{D}_k$ has been determined in [8] for $k \leq 6$ and in [10] for $k \geq 7$. For $k \geq 3$, let $\mathcal{D}_k^*$ and $\mathcal{D}_k^+$ be as follows:

$$\begin{aligned} \mathcal{D}_k^* &= \{(\theta_{k-2}, 0), (\theta_{k-3}, 2 \cdot 3^{k-2}), (\theta_{k-2}, 2 \cdot 3^{k-2}), (\theta_{k-2} + 3^{k-2}, 3^{k-2})\}, \\ \mathcal{D}_k^+ &= \mathcal{D}_k \setminus \mathcal{D}_k^*, \end{aligned}$$

where $\theta_j = (3^{j+1} - 1)/2$. It is known that $\mathcal{D}_k^*$ is included in $\mathcal{D}_k$ and that C is extendable if $(\Phi_0, \Phi_1) \in \mathcal{D}_k^*$ ([8]). Hence it suffices to investigate the extendability of C for $(\Phi_0, \Phi_1) \in \mathcal{D}_k^+$. It is also known that $\mathcal{D}_3^+ = \{(4, 3)\}$ and that an $[n, 3, d]_3$ code with diversity (4,3) is extendable if and only if $\Phi_e > 0$ ([8]). The necessary and sufficient conditions for the extendability of C with $(\Phi_0, \Phi_1) \in \mathcal{D}_k^+$ are given in [10] for $k = 4$ and in [11] for $k = 5$. In this paper, we give the necessary and sufficient conditions for the extendability of an $[n, k, d]_3$ code with $\gcd(3, d) = 1$, general $k \geq 5$, whose diversity is in $\mathcal{D}_k^+$. It is expected that our results would be applicable to (l, s)-extension of ternary linear codes (e.g. see [14]). We also survey the known results about the extendability of ternary linear codes before giving our main theorem (Theorem 4.5).

2. Geometric preliminaries. We denote by $\mathrm{PG}(r, q)$ the projective geometry of dimension r over $\mathrm{GF}(q)$. A *j-flat* is a projective subspace of dimension j in $\mathrm{PG}(r, q)$. 0-flats, 1-flats, 2-flats, 3-flats and $(r-1)$-flats are called *points, lines, planes, solids* and *hyperplanes* respectively as usual. We denote by $\mathcal{F}_j$ the set of j-flats of $\mathrm{PG}(r, q)$ and denote by θ_j the number of points in a j-flat, i.e. $\theta_j = |\mathrm{PG}(j, q)| = (q^{j+1} - 1)/(q - 1)$, where $|T|$ denotes the number of elements in T for a given set T. We set $\theta_j = 0$ when $j < 0$ for convenience.

For an $[n, k, d]_q$ code C with a generator matrix G, the columns of G can be considered as a multiset of n points in $\Sigma = \mathrm{PG}(k-1, q)$ denoted by $\bar{G}$. An

i-point is a point of Σ which has multiplicity i in $\bar{G}$. Let Σ_i be the set of i-points in Σ. For any subset S of Σ we define *the multiplicity of S with respect to C* as

$$m_C(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap \Sigma_i|,$$

where $\gamma_0 = \max\{i \mid \text{an } i\text{-point exists}\}$.

Then we obtain the partition $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_{\gamma_0}$ such that

$$\begin{aligned} n &= m_C(\Sigma), \\ n - d &= \max\{m_C(\pi) \mid \pi \in \mathcal{F}_{k-2}\}. \end{aligned}$$

Conversely such a partition of Σ as above gives an $[n, k, d]_q$ code in the natural manner. Since $(n+1) - (d+1) = n - d$, we get the following.

Lemma 2.1. *C is extendable if and only if there exists a point $P \in \Sigma$ such that $m_C(\pi) < n - d$ for all hyperplanes π through P.*

Let Σ^* be the dual space of Σ (considering $\mathcal{F}_{k-2}$ as the set of points of Σ^*). Then Lemma 2.1 is equivalent to the following:

Lemma 2.2. *C is extendable if and only if there exists a hyperplane Π of Σ^* such that*

$$\Pi \subset \{\pi \in \mathcal{F}_{k-2} \mid m_C(\pi) < n - d\}.$$

Now, let C be an $[n, k, d]_3$ code with diversity (Φ_0, Φ_1), $\gcd(3, d) = 1$, $k \geq 3$, and let $\mathcal{F}_j^*$ be the set of j-flats of Σ^*, i.e., $\mathcal{F}_j^* = \mathcal{F}_{k-2-j}, 0 \leq j \leq k-2$. We define F_0, F_1, F_{e}, F and $\bar{F}$ as follows:

$$\begin{aligned} F_0 &= \{\pi \in \mathcal{F}_0^* \mid m_C(\pi) \equiv n \pmod 3\}, \\ F_1 &= \{\pi \in \mathcal{F}_0^* \mid m_C(\pi) \not\equiv n,\ n-d \pmod 3\}, \\ F_{\mathrm{e}} &= \{\pi \in \mathcal{F}_0^* \mid m_C(\pi) < n-d,\ m_C(\pi) \equiv n-d \pmod 3\}, \\ F &= F_0 \cup F_1, \quad \bar{F} = F \cup F_{\mathrm{e}}. \end{aligned}$$

Then we have $\Phi_0 = |F_0|, \Phi_1 = |F_1|, \Phi_{\mathrm{e}} = |F_{\mathrm{e}}|$ since $|\{\pi \in \mathcal{F}_{k-2} \mid m_C(\pi) = i\}| = A_{n-i}/(q-1)$. Lemma 2.2 implies the following:

Lemma 2.3. *C is extendable if and only if $\bar{F}$ contains a hyperplane of Σ^*.*

We consider the extendability of C from this geometrical point of view. A t-flat Π of Σ^* with $|\Pi \cap F_0| = i$, $|\Pi \cap F_1| = j$ is called an $(i,j)_t$ *flat.* A $(1,0)_0$ flat is just a point of F_0. An $(i,j)_1$ flat, an $(i,j)_2$ flat and an $(i,j)_3$ flat are called an (i,j)-*line*, an (i,j)-*plane* and an (i,j)-*solid* respectively.

Let Λ_1 be the set of all possible (i,j) for which an (i,j)-line exists in $\mathcal{F}_1^*$. Then we have

$$\Lambda_1 = \{(1,0),(0,2),(2,1),(1,3),(4,0)\},$$

see [8]. Assume $2 \le t \le k-1$ and let $\Pi \in \mathcal{F}_t^*$. Denote by $c_{i,j}^{(t)}$ the number of $(i,j)_{t-1}$ flats in Π and let $\varphi_s^{(t)} = |\Pi \cap F_s|$, $s = 0,1$. The pair $(\varphi_0^{(t)}, \varphi_1^{(t)})$ is called the *diversity of* Π and the list of $c_{i,j}^{(t)}$'s is called its *spectrum.* Let Λ_t be the set of all possible $(\varphi_0^{(t)}, \varphi_1^{(t)})$. Λ_t and the corresponding spectra are determined as in Table 1 for $t = 2$ and as in Table 2 for $t = 3$.

Table 1

$\varphi_0^{(2)}$	$\varphi_1^{(2)}$	$c_{1,0}^{(2)}$	$c_{0,2}^{(2)}$	$c_{2,1}^{(2)}$	$c_{1,3}^{(2)}$	$c_{4,0}^{(2)}$
4	0	12	0	0	0	1
1	6	2	9	0	2	0
4	3	4	3	6	0	0
4	6	0	3	6	4	0
7	3	1	0	9	1	2
4	9	0	0	0	12	1
13	0	0	0	0	0	13

Table 2

$\varphi_0^{(3)}$	$\varphi_1^{(3)}$	$c_{4,0}^{(3)}$	$c_{1,6}^{(3)}$	$c_{4,3}^{(3)}$	$c_{4,6}^{(3)}$	$c_{7,3}^{(3)}$	$c_{4,9}^{(3)}$	$c_{13,0}^{(3)}$
13	0	39	0	0	0	0	0	1
4	18	2	36	0	0	0	2	0
13	9	4	3	27	0	6	0	0
10	15	0	10	15	15	0	0	0
16	12	0	0	12	12	16	0	0
13	18	0	3	0	27	6	4	0
22	9	1	0	0	0	36	1	2
13	27	0	0	0	0	0	39	1
40	0	0	0	0	0	0	0	40

For $t \geq 2$ we set

$\Lambda_t^- = \{(\theta_{t-1}, 0), (\theta_{t-2}, 2\cdot 3^{t-1}), (\theta_{t-1}, 2\cdot 3^{t-1}), (\theta_{t-1}+3^{t-1}, 3^{t-1}), (\theta_{t-1}, 3^t), (\theta_t, 0)\}.$

It is known that Λ_t^- is included in Λ_t for all $t \geq 2$ ([8]).

Lemma 2.4. ([8]). *For $t \geq 2$, the spectrum corresponding to each diversity in Λ_t^- is uniquely determined as follows:*
(1) $(c^{(t)}_{\theta_{t-2},0}, c^{(t)}_{\theta_{t-1},0}) = (\theta_t - 1, 1)$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, 0)$;
(2) $(c^{(t)}_{\theta_{t-2},0}, c^{(t)}_{\theta_{t-3},2\cdot 3^{t-2}}, c^{(t)}_{\theta_{t-2},3^{t-1}}) = (2, \theta_t - \theta_1, 2)$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-2}, 2\cdot 3^{t-1})$;
(3) $(c^{(t)}_{\theta_{t-3},2\cdot 3^{t-2}}, c^{(t)}_{\theta_{t-2},2\cdot 3^{t-2}}, c^{(t)}_{\theta_{t-2}+3^{t-2},3^{t-2}}, c^{(t)}_{\theta_{t-2},3^{t-1}}) = (3, \theta_t - \theta_2, 6, 4)$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, 2\cdot 3^{t-1})$;
(4) $(c^{(t)}_{\theta_{t-2},0}, c^{(t)}_{\theta_{t-2}+3^{t-2},3^{t-2}}, c^{(t)}_{\theta_{t-2},3^{t-1}}, c^{(t)}_{\theta_{t-1},0}) = (1, \theta_t - \theta_1, 1, 2)$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} + 3^{t-1}, 3^{t-1})$;
(5) $(c^{(t)}_{\theta_{t-2},3^{t-1}}, c^{(t)}_{\theta_{t-1},0}) = (\theta_t - 1, 1)$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, 3^t)$;
(6) $c^{(t)}_{\theta_{t-1},0} = \theta_t$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_t, 0)$.

Set $\Lambda_t^+ = \Lambda_t \setminus \Lambda_t^-$. The diversities in Λ_t^+ and the corresponding spectra for $t \geq 4$ are determined as follows.

Lemma 2.5 ([10]). (1) *When t is odd (≥ 5):*
$\Lambda_t^+ = \{(\theta_{t-1}, 3^{t-1})\} \cup \{(\theta_{t-1} - 3^{T+1+s}, \theta_{t-1} + \theta_{T+s} + 1), (\theta_{t-1} + 3^{T+1+s}, \theta_{t-1} - \theta_{T+s}) \mid 0 \leq s \leq T\} \cup \{(\theta_{t-1}, \theta_{t-1} - \theta_{T+s}), (\theta_{t-1}, \theta_{t-1} + \theta_{T+s} + 1) \mid 1 \leq i \leq T\}$,
where $T = (t-3)/2$. The spectrum corresponding to each diversity is uniquely determined as follows:
(A-1) $c^{(t)}_{\theta_{t-2}-3^{T+1},\theta_{t-2}+\theta_T+1} = \theta_{t-1} - 3^{T+1}$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_T} = c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_T+1} = \theta_{t-1} + \theta_T + 1$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1)$;
(A-2) $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_T} = c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_T+1} = \theta_{t-1} - \theta_T$, $c^{(t)}_{\theta_{t-2}+3^{T+1},\theta_{t-2}-\theta_T} = \theta_{t-1} + 3^{T+1}$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} + 3^{T+1}, \theta_{t-1} - \theta_T)$;
(A-3) $(c^{(t)}_{\theta_{t-2},0}, c^{(t)}_{\theta_{t-3},2\cdot 3^{t-2}}, c^{(t)}_{\theta_{t-2},3^{t-2}}, c^{(t)}_{\theta_{t-2}+3^{t-2},3^{t-2}}) = (4, 3, \theta_t - \theta_2, 6)$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, 3^{t-1})$;
(A-4) $c^{(t)}_{\theta_{t-2}-3^{T+1+s},\theta_{t-2}+\theta_{T+s}+1} = \theta_{t-1-2s} - 3^{T+1-s}$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{T+s}} = c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{T+s}+1} = \theta_{t-1-2s} + \theta_{T-s} + 1$, $c^{(t)}_{\theta_{t-2}-3^{T+s},\theta_{t-2}+\theta_{T-1+s}+1} = \theta_t - \theta_{t-2s}$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} - 3^{T+1+s}, \theta_{t-1} + \theta_{T+s} + 1)$, $1 \leq s \leq T$;

(*A*-5) $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{T+s}} = c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{T+s}+1} = \theta_{t-1-2s} - \theta_{T-s}$, $c^{(t)}_{\theta_{t-2}+3^{T+1+s},\theta_{t-2}-\theta_{T+s}} = \theta_{t-1-2s} + 3^{T+1-s}$, $c^{(t)}_{\theta_{t-2}+3^{T+s},\theta_{t-2}-\theta_{T-1+s}} = \theta_t - \theta_{t-2s}$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1} + 3^{T+1+s}, \theta_{t-1} - \theta_{T+s})$, $1 \le s \le T$;

(*A*-6) $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{T+s}} = \theta_{t-2s}$, $c^{(t)}_{\theta_{t-2}-3^{T+s},\theta_{t-2}+\theta_{T-1+s}+1} = \theta_{t-2s} - \theta_{T+1-s}$, $c^{(t)}_{\theta_{t-2}+3^{T+s},\theta_{t-2}-\theta_{T-1+s}} = \theta_{t-2s} + \theta_{T+1-s} + 1$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{T-1+s}} = \theta_t - \theta_{t+1-2s}$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1}, \theta_{t-1} - \theta_{T+s})$, $1 \le s \le T$;

(*A*-7) $c^{(t)}_{\theta_{t-2}-3^{T+s},\theta_{t-2}+\theta_{T-1+s}+1} = \theta_{t-2s} - \theta_{T+1-s}$, $c^{(t)}_{\theta_{t-2}+3^{T+s},\theta_{t-2}-\theta_{T-1+s}} = \theta_{t-2s} + \theta_{T+1-s} + 1$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{T+s}+1} = \theta_{t-2s}$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{T-1+s}+1} = \theta_t - \theta_{t+1-2s}$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1}, \theta_{t-1} + \theta_{T+s} + 1)$, $1 \le s \le T$.

(2) *When t is even* (≥ 4)*:*

$\Lambda^+_t = \{(\theta_{t-1}, 3^{t-1})\} \cup \{(\theta_{t-1}, \theta_{t-1} - \theta_{U+1+s}), (\theta_{t-1}, \theta_{t-1} + \theta_{U+1+s} + 1) \mid 0 \le s \le U\} \cup \{(\theta_{t-1} - 3^{U+1+s}, \theta_{t-1} + \theta_{U+s} + 1), (\theta_{t-1} + 3^{U+1+s}, \theta_{t-1} - \theta_{U+s}) \mid 1 \le s \le U + 1\}$,

where $U = (t-4)/2$. The spectrum corresponding to each diversity is uniquely determined as follows:

(*B*-1) $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{U+1}} = \theta_{t-1}$, $c^{(t)}_{\theta_{t-2}-3^{U+1},\theta_{t-2}+\theta_{U}+1} = \theta_{t-1} - \theta_{U+1}$, $c^{(t)}_{\theta_{t-2}+3^{U+1},\theta_{t-2}-\theta_U} = \theta_{t-1} + \theta_{U+1} + 1$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1}, \theta_{t-1} - \theta_{U+1})$;

(*B*-2) $c^{(t)}_{\theta_{t-2}-3^{U+1},\theta_{t-2}+\theta_U+1} = \theta_{t-1} - \theta_{U+1}$, $c^{(t)}_{\theta_{t-2}+3^{U+1},\theta_{t-2}-\theta_U} = \theta_{t-1} + \theta_{U+1} + 1$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{U+1}+1} = \theta_{t-1}$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1}, \theta_{t-1} + \theta_{U+1} + 1)$;

(*B*-3) $(c^{(t)}_{\theta_{t-2},0}, c^{(t)}_{\theta_{t-3},2\cdot 3^{t-2}}, c^{(t)}_{\theta_{t-2},3^{t-2}}, c^{(t)}_{\theta_{t-2}+3^{t-2},3^{t-2}}) = (4, 3, \theta_t - \theta_2, 6)$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1}, 3^{t-1})$;

(*B*-4) $c^{(t)}_{\theta_{t-2}-3^{U+1+s},\theta_{t-2}+\theta_{U+s}+1} = \theta_{t-2s} - 3^{U+2-s}$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{U+s}} = c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{U+s}+1} = \theta_{t-2s} + \theta_{U+1-s} + 1$, $c^{(t)}_{\theta_{t-2}-3^{U+s},\theta_{t-2}+\theta_{U-1+s}+1} = \theta_t - \theta_{t+1-2s}$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1} - 3^{U+1+s}, \theta_{t-1} + \theta_{U+s} + 1)$, $1 \le s \le U + 1$;

(*B*-5) $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{U+s}} = c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{U+s}+1} = \theta_{t-2s} - \theta_{U+1-s}$, $c^{(t)}_{\theta_{t-2}+3^{U+1+s},\theta_{t-2}-\theta_{U+s}} = \theta_{t-2s} + 3^{U+2-s}$, $c^{(t)}_{\theta_{t-2}+3^{U+s},\theta_{t-2}-\theta_{U-1+s}} = \theta_t - \theta_{t+1-2s}$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1} + 3^{U+1+s}, \theta_{t-1} - \theta_{U+s})$, $1 \le s \le U + 1$;

$(B\text{-}6)$ $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{U+1+s}} = \theta_{t-1-2s},\ c^{(t)}_{\theta_{t-2}-3^{U+1+s},\theta_{t-2}+\theta_{U+s}+1} = \theta_{t-1-2s} - \theta_{U+1-s},$ $c^{(t)}_{\theta_{t-2}+3^{U+1+s},\theta_{t-2}-\theta_{U+s}} = \theta_{t-1-2s} + \theta_{U+1-s} + 1,\ c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{U+s}} = \theta_t - \theta_{t-2s}$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1}, \theta_{t-1} - \theta_{U+1+s}),\ 1 \le s \le U;$

$(B\text{-}7)$ $c^{(t)}_{\theta_{t-2}-3^{U+1+s},\theta_{t-2}+\theta_{U+s}+1} = \theta_{t-1-2s} - \theta_{U+1-s},\ c^{(t)}_{\theta_{t-2}+3^{U+1+s},\theta_{t-2}-\theta_{U+s}} = \theta_{t-1-2s}$ $+ \theta_{U+1-s} + 1,\ c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{U+1+s}+1} = \theta_{t-1-2s},\ c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{U+s}+1} = \theta_t - \theta_{t-2s}$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1}, \theta_{t-1} + \theta_{U+1+s} + 1),\ 1 \le s \le U.$

3. Characterizations of $(i,j)_t$ flats in Σ^*. Let Π be a t-flat in Σ^*. An s-flat S in Π is called the *axis of* Π *of type* (a,b) if every hyperplane of Π not containing S has the same diversity (a,b) and if there is no hyperplane of Π through S whose diversity is (a,b). Then the spectrum of Π satisfies $c^{(t)}_{a,b} = \theta_t - \theta_{t-1-s}$ and the axis is unique if it exists. The axis is helpful to characterize the geometrical structure of Π.

The geometrical structure of Π whose diversity is in Λ^-_t can be seen as the following lemma by means of the axis of Π. As for the type (3) of Lemma 2.4 for $t = 2$, see [8].

Lemma 3.1 ([9])**.** *Let* Π *be a* t*-flat in* Σ^*.
(1) *For* $(i,j) = (\theta_{t-1}, 0)$ *or* $(\theta_{t-1}, 3^t)$, $t \ge 2$, Π *is an* $(i,j)_t$ *flat if and only if* Π *contains a* $(\theta_{t-1}, 0)_{t-1}$ *flat which is the axis of type* $((i-1)/3, j/3)$.
(2) *For* $(i,j) = (\theta_{t-2}, 2\cdot 3^{t-1})$ *or* $(\theta_{t-1} + 3^{t-1}, 3^{t-1})$, $t \ge 2$, Π *is an* $(i,j)_t$ *flat if and only if* Π *contains a* $(\theta_{t-2}, 0)_{t-2}$ *flat which is the axis of type* $((i-1)/3, j/3)$.
(3) *For* $(i,j) = (\theta_{t-1}, 3^{t-1})$ *or* $(\theta_{t-1}, 2\cdot 3^{t-1})$, $t \ge 3$, Π *is an* $(i,j)_t$ *flat if and only if* Π *contains a* $(\theta_{t-3}, 0)_{t-3}$ *flat which is the axis of type* $((i-1)/3, j/3)$.

Lemma 3.2. *Let* Π *be a* 4*-flat in* Σ^* *and let* $(i,j) \in \{(31,45),(49,36)\}$. *Then* Π *is an* $(i,j)_4$ *flat if and only if* Π *contains a point of* F_0 *which is the axis of type* $((i-1)/3, j/3)$.

P r o o f. We prove only for $(i,j) = (31,45)$. One can prove it for $(i,j) = (49,36)$ in a similar way.

"only if" part: Assume that Π is a $(31,45)_4$ flat. By Lemma 2.5 (B-4), the spectrum of Π is $(c^{(4)}_{4,18}, c^{(4)}_{13,9}, c^{(4)}_{10,15}, c^{(4)}_{13,18})$ =(10,15,81,15). There are exactly two $(13,9)$-solids π_1, π_2 and two $(13,18)$-solids π_3, π_4 through a fixed $(7,3)$-plane δ. From (3) of Lemma 3.1, a $(13,9)$-solid contains a point of F_0 which is the axis of type $(4,3)$, and a $(13,18)$-solid contains a point of F_0 which is the axis of type $(4,6)$. Since δ contains a point P of F_0 which is the axis of type (2,1) by (2) of Lemma 3.1, the axis of π_i $(1 \le i \le 4)$ coincides with P. Let Δ be a solid not containing P. Δ meets π_1, π_2 in a $(4,3)$-plane and π_3, π_4 in a $(4,6)$-plane. Therefore Δ is just a $(10,15)$-solid. There is no $(10,15)$-solid through P since $c^{(4)}_{10,15} = 81$. Hence P is the axis of Π of type $(10,15)$.

"if" part: Assume that a 4-flat Π contains a point P of F_0 which is the axis of type (10,15). From the definition of the axis, all the 4-flats not containing P are (10,15)-solids. Hence the number of (10,15)-solids is at least 81. Thus Π is just a $(31,45)_4$ flat. □

Lemma 3.3. *Let* Π *be a* 5*-flat in* Σ^**, and let* $(i,j) \in \{(121,108),(121,135)\}$*. Then* Π *is an* $(i,j)_5$ *flat if and only if* Π *contains a point of* F_0 *which is the axis of type* $((i-1)/3, j/3)$*.*

Proof. We prove this only for $(i,j) = (121,108)$. The other case is proved similarly.

"only if" part: Assume that Π is a $(121,108)_5$ flat. By Lemma 2.5 (A-6), the spectrum of Π is $(c^{(5)}_{40,27}, c^{(5)}_{31,45}, c^{(5)}_{40,36}, c^{(5)}_{49,36}) = (40,36,243,45)$. There are exactly two $(31,45)_4$ flats π_1, π_2 and two $(49,36)_4$ flats π_3, π_4 through a fixed $(13,18)$-solid δ. From Lemma 3.2, a $(31,45)_4$ flat contains a point of F_0 which is the axis of type (10,15), and a $(49,36)_4$ flat contains a point of F_0 which is the axis of type $(16,12)$. Since δ contains a point P of F_0 which is the axis of type $(4,6)$ by (3) of Lemma 3.1, the axis of π_i $(1 \le i \le 4)$ coincides with P. Indeed, any solid Δ in π_1 not containing P is a $(10,15)$-solid, for $\Delta \cap \delta$ is a $(4,6)$-plane and there is exactly one $(13,18)$-solid through a fixed $(4,6)$-plane in π_1. Let π be a 4-flat not containing P. π meets π_1, π_2 in a $(10,15)$-solid and π_3, π_4 in a $(16,12)$-solid. Therefore π is just a $(40,36)_4$ flat. There is no $(40,36)_4$ flat through P since $c^{(5)}_{40,36} = 324$. Hence P is the axis of Π of type $(40,36)$. The "if" part is similar to the one in Lemma 3.2. □

Lemma 3.4. (1) *Let* Π *be a* t*-flat in* Σ^* *with even* $t \ge 4$*, and let* $(i,j) \in \{(\theta_{t-1} - 3^{U+1+s}, \theta_{t-1} + \theta_{U+s} + 1), (\theta_{t-1} + 3^{U+1+s}, \theta_{t-1} - \theta_{U+s}) \mid 1 \le s \le U+1\}$, $U = (t-4)/2$*. Then* Π *is an* $(i,j)_t$ *flat if and only if* Π *contains a* $(\theta_{2s-2}, 0)_{2s-2}$ *flat which is the axis of type* $((i-1)/3, j/3)$*.*
(2) *Let* Π *be a* t*-flat in* Σ^* *with odd* $t \ge 5$*, and let* $(i,j) \in \{(\theta_{t-1}, \theta_{t-1} - \theta_{T+s}), (\theta_{t-1}, \theta_{t-1} + \theta_{T+s} + 1) \mid 1 \le s \le T\}$, $T = (t-3)/2$*. Then* Π *is an* $(i,j)_t$ *flat if and only if* Π *contains a* $(\theta_{2s-2}, 0)_{2s-2}$ *flat which is the axis of type* $((i-1)/3, j/3)$*.*

Proof. We prove this only for $(\theta_{t-1} - 3^{U+1+s}, \theta_{t-1} + \theta_{U+s} + 1)_t$ flat of (1) and for $(\theta_{t-1}, \theta_{t-1} - \theta_{T+s})_t$ flat of (2). The other cases are proved similarly.

"only if" part: We prove this part by induction on t (≥ 4). First, (1) holds for $t = 4$ and (2) holds for $t = 5$. Next, we assume (1) for $t-1$ and (2) for $t-2$ to prove (2) for t.

Claim 1. Let π be a $(i,j)_{t-1}$ flat in Σ^* with $t \geq 5$, and let $(i,j) \in \{(\theta_{t-2} - 3^{T+s}, \theta_{t-2} + \theta_{T-1+s} + 1), (\theta_{t-2} + 3^{T+s}, \theta_{t-2} - \theta_{T-1+s})\}$. Then the axis of π coincides with the axis of a $(\theta_{t-3}, \theta_{t-3} + \theta_{T-1+s} + 1)_{t-2}$ flat in π.

To see Claim 1, let π be a $(\theta_{t-2} - 3^{T+s}, \theta_{t-2} + \theta_{T-1+s} + 1)_{t-1}$ flat. The spectrum of π is $(c^{(t-1)}_{\theta_{t-3}-3^{T+s},\theta_{t-3}+\theta_{T-1+s}+1}, c^{(t-1)}_{\theta_{t-3},\theta_{t-3}-\theta_{T-1+s}}, c^{(t-1)}_{\theta_{t-3},\theta_{t-3}+\theta_{T-1+s}+1}, c^{(t-1)}_{\theta_{t-3}-3^{T-1+s},\theta_{t-3}+\theta_{T-2+s}+1}) = (\theta_{t-1-2s} - 3^{T+1-s}, \theta_{t-1-2s} + \theta_{T-s} + 1, \theta_{t-1-2s} + \theta_{T-s}+1, \theta_{t-1}-\theta_{t-2s})$ by Lemma 2.5 (B-4) for $t-1$. From the induction hypothesis for $t-2$, a $(\theta_{t-3}, \theta_{t-3}+\theta_{T-1+s}+1)_{t-2}$ flat Δ contains a $(\theta_{2s-2}, 0)_{2s-2}$ flat ϖ which is the axis of type $(\theta_{t-4}, \theta_{t-4} + \theta_{T-2+s} + 1)$. A $(t-3)$-flat in Δ not containing ϖ is a $(\theta_{t-4}, \theta_{t-4}+\theta_{T-2+s}+1)_{t-3}$ flat, say δ. A $(t-2)$-flat Δ' in π containing δ is a $(\theta_{t-3}, \theta_{t-3}+\theta_{T-1+s}+1)_{t-2}$ flat or a $(\theta_{t-3}-3^{T-1+s}, \theta_{t-3}+\theta_{T-2+s}+1)_{t-2}$ flat by the spectrum of π. However, there is only one $(\theta_{t-3}, \theta_{t-3}+\theta_{T-1+s}+1)_{t-2}$ flat through a fixed $(\theta_{t-4}, \theta_{t-4} + \theta_{T-2+s} + 1)_{t-3}$ flat. So Δ' is just a $(\theta_{t-3} - 3^{T-1+s}, \theta_{t-3} + \theta_{T-2+s} + 1)_{t-2}$ flat. Since the number of $(t-3)$-flats in Δ not containing ϖ is $\theta_{t-2} - \theta_{t-2s-1}$, the number of $(\theta_{t-3} - 3^{T-1+s}, \theta_{t-3} + \theta_{T-2+s} + 1)_{t-2}$ flats in π not containing ϖ is $\theta_{t-1} - \theta_{t-2s}$. From the spectrum of π and the definition of the axis, the axis ϖ of Δ coincides with the axis of π. Similarly, the axis of a $(\theta_{t-3}, \theta_{t-3} + \theta_{T-1+s} + 1)_{t-2}$ flat coincides with the axis of a $(\theta_{t-2} + 3^{T+s}, \theta_{t-2} - \theta_{T-1+s})_{t-1}$ flat.

For $t \geq 7$, assume that Π is a $(\theta_{t-1}, \theta_{t-1} - \theta_{T+s})_t$ flat. From the spectrum of Π (see Lemma 2.5 (A-6)), there are exactly two $(\theta_{t-2} - 3^{T+s}, \theta_{t-2} + \theta_{T-1+s} + 1)_{t-1}$ flats π'_1, π'_2 and two $(\theta_{t-2} + 3^{T+s}, \theta_{t-2} - \theta_{T-1+s})_{t-1}$ flats π'_3, π'_4 through a fixed $(\theta_{t-3}, \theta_{t-3} + \theta_{T-1+s} + 1)_{t-2}$ flat Δ'. By Claim 1, the axis of Δ', say δ', coincides with the axes of π'_1–π'_4. Let π' be a $(t-1)$-flat not containing δ'. Since π' meets π_1, π_2 in a $(\theta_{t-3} - 3^{T-1+s}, \theta_{t-3} + \theta_{T-2+s} + 1)_{t-2}$ flat and π_3, π_4 in a $(\theta_{t-3} + 3^{T-1+s}, \theta_{t-3} - \theta_{T-2+s})_{t-2}$ flat, π' is just a $(\theta_{t-2}, \theta_{t-2} - \theta_{T-1+s})_{t-1}$ flat. Hence Π contains a $(\theta_{2s-2}, 0)_{2s-2}$ flat which is the axis of type $((i-1)/3, j/3)$.

Finally, we assume (2) for $t-1$ and (1) for $t-2$ to prove (1) for t.

Claim 2. Let π be a $(i,j)_{t-1}$ flat in Σ^* with $t \geq 5$, and let $(i,j) \in \{(\theta_{t-2}, \theta_{t-2} - \theta_{U+s}), (\theta_{t-2}, \theta_{t-2} + \theta_{U+s} + 1)\}$. Then the axis of π coincides with

the axis of a $(\theta_{t-3}+3^{U+s},\theta_{t-3}-\theta_{T-1+s})_{t-2}$ flat in π.
This claim is proved similarly to Claim 1.

For $t\geq 6$, assume that Π is a $(\theta_{t-1}-3^{U+1+s},\theta_{t-1}+\theta_{U+s}+1)_t$ flat. From the spectrum of Π (see Lemma 2.5 (B-4)), there are exactly two $(\theta_{t-2},\theta_{t-2}-\theta_{U+s})_{t-1}$ flats π'_1,π'_2 and two $(\theta_{t-2},\theta_{t-2}+\theta_{U+s}+1)_{t-1}$ flats π'_3,π'_4 through a fixed $(\theta_{t-3}+3^{U+s},\theta_{t-3}-\theta_{T-1+s})_{t-2}$ flat Δ'. By Claim 2, the axis of Δ', say δ', coincides with the axes of π'_1–π'_4. Let π' be a $(t-1)$-flat not containing δ'. Since π' meets π'_1,π'_2 in a $(\theta_{t-3},\theta_{t-3}-\theta_{U+s-1})_{t-2}$ flat and π'_3,π'_4 in a $(\theta_{t-3},\theta_{t-3}+\theta_{U+s-1}+1)_{t-2}$ flat, π' is just a $(\theta_{t-2}-3^{U+s},\theta_{t-2}+\theta_{U+s-1}+1)_{t-1}$ flat. Hence Π contains a $(\theta_{2s-2},0)_{2s-2}$ flat which is the axis of type $((i-1)/3,j/3)$.

"if" part: We prove only for the axis of type $(\theta_{t-2}-3^{U+s},\theta_{t-2}+\theta_{U+s-1}+1)$ of (1). The others are proved similarly. Assume that a t-flat Π contains a $(\theta_{2s-2},0)_{2s-2}$ flat δ which is the axis of type $(\theta_{t-2}-3^{U+s},\theta_{t-2}+\theta_{U+s-1}+1)$. From the definition of the axis, all the $(t-1)$-flats not containing δ are $(\theta_{t-2}-3^{U+s},\theta_{t-2}+\theta_{U+s-1}+1)_{t-1}$ flats. Hence the number of $(\theta_{t-2}-3^{U+s},\theta_{t-2}+\theta_{U+s-1}+1)_{t-1}$ flats are at least $\theta_t-\theta_{t-2s}$. Thus Π is just a $(\theta_{t-1}-3^{U+1+s},\theta_{t-1}+\theta_{U+s}+1)_t$ flat by Lemma 2.5. □

Lemma 3.5. (1) *Let Π be a t-flat in Σ^* with even $t\geq 4$, and let $(i,j)\in\{(\theta_{t-1},\theta_{t-1}-\theta_{U+s}),(\theta_{t-1},\theta_{t-1}+\theta_{U+s}+1)\mid 1\leq s\leq U\}$ $U=(t-4)/2$. Then Π is an $(i,j)_t$ flat if and only if Π contains a $(\theta_{2s-1},0)_{2s-1}$ flat which is the axis of type $((i-1)/3,j/3)$.*
(2) *Let Π be a t-flat in Σ^* with odd $t\geq 5$, and let $(i,j)\in\{(\theta_{t-1}-3^{T+1+s},\theta_{t-1}+\theta_{T+s}+1),(\theta_{t-1}+3^{T+1+s},\theta_{t-1}-\theta_{T+s})\mid 1\leq s\leq T\}$, $T=(t-3)/2$. Then Π is an $(i,j)_t$ flat if and only if Π contains a $(\theta_{2s-1},0)_{2s-1}$ flat which is the axis of type $((i-1)/3,j/3)$.*

The above lemma can be proved similarly to Lemma 3.4.

Lemma 3.6. (1) *Let Π be a t-flat in Σ^* with even $t\geq 4$ and $U=(t-4)/2$. Then Π is a $(\theta_{t-1},\theta_{t-1}-\theta_{U+1})_t$ flat if and only if Π contains four $(t-1)$-flats $\pi_1,\cdots,\pi_4$ through a fixed $(\theta_{t-3},\theta_{t-3}-\theta_{U+1})_{t-2}$ flat Δ such that Δ contains a $(4,0)$-line $l=\{P_1,P_2,P_3,P_4\}$ which is the axis of type $(\theta_{t-4},\theta_{t-4}-\theta_U)$ and that P_i is the axis of π_i of type $(\theta_{t-3},\theta_{t-3}-\theta_U)$ for $1\leq i\leq 4$.*
(2) *For a t-flat Π in Σ^* with odd $t\geq 5$ and $T=(t-3)/2$, Π is a $(\theta_{t-1}+3^{T+1},\theta_{t-1}-\theta_T)_t$ flat if and only if Π contains four $(t-1)$-flats $\pi_1\cdots\pi_4$ through a fixed $(\theta_{t-3}+3^{T+1},\theta_{t-3}-\theta_T)_{t-2}$ flat Δ such that Δ contains a $(4,0)$-line $l=\{P_1,P_2,P_3,P_4\}$ which is the axis of type $(\theta_{t-4}+3^T,\theta_{t-4}-\theta_{T-1})$ and that P_i is the axis of π_i of type $(\theta_{t-3}+3^T,\theta_{t-3}-\theta_{T-1})$ for $1\leq i\leq 4$.*

P r o o f. "only if" part: We prove this part by induction on $t \geq 4$. Assume that Π is a $(49, 36)_4$ flat. The spectrum of Π is $(c^{(4)}_{13,9}, c^{(4)}_{10,15}, c^{(4)}_{16,12}) = (40, 36, 45)$ from Lemma 2.5 (A-1). There are exactly four (13,9)-solids $\pi_1, \cdots, \pi_4$ through a fixed $(4, 0)$-plane Δ in Π. A $(13, 9)$-solid contains a point $P_i \in F_0$ which is the axis of type $(4, 3)$ by (3) of Lemma 3.1. And a $(4, 0)$-plane contains a $(4, 0)$-line l which is the axis of type $(1, 0)$ by (1) of Lemma 3.1. Therefore the axis $P_i \in F_0$ of π_i is on l. Suppose $P_1 = P_2$. Then there are two $(1, 6)$-planes $\Delta_1 \subset \pi_1, \Delta_2 \subset \pi_2$ through a fixed $(1, 0)$-line $l' \subset \Delta$ containing $P_1 (= P_2)$. $\pi = \langle \Delta_1, \Delta_2 \rangle$ is a $(13, 9)$-solid or a $(10, 15)$-solid since a $(16, 12)$-solid contains no $(1, 6)$-plane from Table 2. In a $(10, 15)$-solid, there is only one $(1, 6)$-plane through a fixed $(1, 0)$-line, so π is not a $(10, 15)$-solid. If π is a $(13, 9)$-solid, then two $(13, 9)$-solids meet in a $(1, 6)$-plane in Π. However there is only one $(13, 9)$-solid through a fixed $(1, 6)$-plane in Π, a contradiction. So $P_1 \neq P_2$, and (1) holds for $t = 4$.

Next, we assume (1) for $t-1$ to prove (2). For $t \geq 5$, assume that Π is a $(\theta_{t-1} + 3^{T+1}, \theta_{t-1} - \theta_T)_t$ flat. By the spectrum of Π (see Lemma 2.5 (A-1)) there are exactly four $(\theta_{t-2} + 3^{T+1}, \theta_{t-2} - \theta_T)_{t-1}$ flats $\pi_1, \cdots, \pi_4$ through a fixed $(\theta_{t-3} + 3^{T+1}, \theta_{t-3} - \theta_T)_{t-2}$ flat Δ. π_i contains a point $P_i \in F_0$ which is the axis of type $(\theta_{t-3} + 3^T, \theta_{t-3} - \theta_{T-1})$ by (1) of Lemma 3.4. And Δ contains a (4,0)-line l which is the axis of type $(\theta_{t-4} + 3^T, \theta_{t-4} - \theta_{T-1})$ by (2) of Lemma 3.5. Therefore the axis $P_i \in F_0$ of π_i is on l. Suppose $P_1 = P_2$. Then there are two $(\theta_{t-3}, \theta_{t-3} - \theta_T)_{t-2}$ flats $\Delta_1 \subset \pi_1, \Delta_2 \subset \pi_2$ through a fixed $(\theta_{t-4} + 3^{T-1+s}, \theta_{t-4} - \theta_{T-2+s})_{t-3}$ flat containing P_1 in Δ. $\langle \Delta_1, \Delta_2 \rangle = \pi$ is a $(\theta_{t-2}, \theta_{t-2} - \theta_T)_{t-1}$ flat or a $(\theta_{t-2} + 3^{T+1}, \theta_{t-2} - \theta_T)_{t-1}$ flat since a $(\theta_{t-2}, \theta_{t-2} + \theta_T + 1)_{t-1}$ flat has no $(\theta_{t-3}, \theta_{t-3} - \theta_T)_{t-2}$ flat. If π is a $(\theta_{t-2}, \theta_{t-2} - \theta_T)_{t-1}$ flat, then the axes of two such $(\theta_{t-3}, \theta_{t-3} - \theta_T)_{t-2}$ flats do not coincide from the induction hypothesis for $t-1$, a contradiction. If π is a $(\theta_{t-2} + 3^{T+1}, \theta_{t-2} - \theta_T)_{t-1}$ flat, then there is only one $(\theta_{t-3}, \theta_{t-3} - \theta_T)_{t-2}$ flat through a fixed $(\theta_{t-4} + 3^{T-1+s}, \theta_{t-4} - \theta_{T-2+s})_{t-3}$ flat in Π, a contradiction. So $P_1 \neq P_2$, and our assertion follows.

For $t-1$, we assume (2) to prove (1). For $t \geq 6$, assume that Π is a $(\theta_{t-1}, \theta_{t-1} - \theta_{U+1})_t$ flat. From Lemma 2.5 (B-1), there are exactly four $(\theta_{t-2}, \theta_{t-2} - \theta_{U+1})_{t-1}$ flats $\pi_1, \cdots, \pi_4$ through a fixed $(\theta_{t-3}, \theta_{t-3} - \theta_{U+1})_{t-2}$ flat Δ. π_i contains a point $P_i \in F_0$ which is the axis of type $(\theta_{t-3}, \theta_{t-3} - \theta_U)$ by (2) of Lemma 3.4. And Δ contains a (4,0)-line l which is the axis of type $(\theta_{t-4}, \theta_{t-4} - \theta_U)$ by (1) of Lemma 3.5. Therefore the axis $P_i \in F_0$ of π_i is on l for $1 \leq i \leq 4$. Suppose $P_1 = P_2$. Then there are two $(\theta_{t-3} + 3^{U+1}, \theta_{t-3} - \theta_U)_{t-2}$ flats $\Delta_1 \subset \pi_1, \Delta_2 \subset \pi_2$ through a fixed $(\theta_{t-4}, \theta_{t-4} - \theta_U)_{t-3}$ flat containing P_1 in Δ. $\langle \Delta_1, \Delta_2 \rangle = \pi$ is a $(\theta_{t-2}, \theta_{t-2} - \theta_{U+1})_{t-1}$ flat or a $(\theta_{t-2} + 3^{U+1}, \theta_{t-2} - \theta_U)_{t-1}$ flat since $(\theta_{t-2} - 3^{U+1}, \theta_{t-2} + \theta_{U+1})_{t-1}$ flat has no $(\theta_{t-3} + 3^{U+1}, \theta_{t-3} - \theta_U)_{t-2}$

flat. If π is a $(\theta_{t-2}+3^{U+1},\theta_{t-2}-\theta_U)_{t-1}$ flat, then the axes of such two $(\theta_{t-3}+3^{U+1},\theta_{t-3}-\theta_U)_{t-2}$ flats do not coincide from the induction hypothesis for $t-1$, a contradiction. If π is a $(\theta_{t-2},\theta_{t-2}-\theta_{U+1})_{t-1}$ flat, then there is only one $(\theta_{t-3}+3^{U+1},\theta_{t-3}-\theta_U)_{t-2}$ flat through a fixed $(\theta_{t-4},\theta_{t-4}-\theta_U)_{t-3}$ flat, a contradiction. So $P_1\neq P_2$, and our assertion follows.

"if" part: Let Π be a t-flat in Σ^* with even $t\geq 4$ and $U=(t-4)/2$. Assume that Π contains four $(t-1)$-flats $\pi_1,\cdots,\pi_4$ through a fixed $(\theta_{t-3},\theta_{t-3}-\theta_{U+1})_{t-2}$ flat Δ such that Δ contains a $(4,0)$-line $l=\{P_1,P_2,P_3,P_4\}$ which is the axis of type $(\theta_{t-4},\theta_{t-4}-\theta_U)$ and that P_i is the axis of π_i of type $(\theta_{t-3},\theta_{t-3}-\theta_U)$. From (1) of Lemma 3.5, a $(t-1)$-flat containing a point $P_i\in F_0$ which the axis of type $(\theta_{t-3},\theta_{t-3}-\theta_U)$ is a $(\theta_{t-2},\theta_{t-2}-\theta_{U+1})_{t-1}$ flat, say π_i. Since there are four $(\theta_{t-2},\theta_{t-2}-\theta_{U+1})_{t-1}$ flats through a fixed $(\theta_{t-3},\theta_{t-3}-\theta_{U+1})_{t-2}$ flat, Π is just a $(\theta_{t-1},\theta_{t-1}-\theta_{U+1})_t$ flat. The other cases are proved similarly. □

The following lemma can be proved similarly to Lemma 3.6.

Lemma 3.7. (1) *Let* Π *be a* t*-flat in* Σ^* *with even* $t\geq 4$*,* $U=(t-4)/2$*. Then* Π *is a* $(\theta_{t-1},\theta_{t-1}+\theta_{U+1}+1)_t$ *flat if and only if* Π *contains four* $(t-1)$*-flats* $\pi_1,\cdots,\pi_4$ *through a fixed* $(\theta_{t-3},\theta_{t-3}+\theta_{U+1}+1)_{t-2}$ *flat* Δ *such that* Δ *contains a* $(4,0)$*-line* $l=\{P_1,P_2,P_3,P_4\}$ *which is the axis of type* $(\theta_{t-4},\theta_{t-4}+\theta_U+1)$ *and that* P_i *is the axis of* π_i *of type* $(\theta_{t-3},\theta_{t-3}+\theta_U+1)$ *for* $1\leq i\leq 4$*.*
(2) *Let* Π *be a* t*-flat in* Σ^* *with odd* $t\geq 5$*,* $T=(t-3)/2$*. Then* Π *is a* $(\theta_{t-1}-3^{T+1},\theta_{t-1}+\theta_T+1)_t$ *flat if and only if* Π *contains four* $(t-1)$*-flats* $\pi_1,\cdots,\pi_4$ *through a fixed* $(\theta_{t-3}-3^{T+1},\theta_{t-3}+\theta_T+1)_{t-2}$ *flat* Δ *such that* Δ *contains a* $(4,0)$*-line* $l=\{P_1,P_2,P_3,P_4\}$ *which is the axis of type* $(\theta_{t-4}-3^T,\theta_{t-4}+\theta_{T-1}+1)$ *and that* P_i *is the axis of* π_i *of type* $(\theta_{t-3}-3^T,\theta_{t-3}+\theta_{T-1}+1)$ *for* $1\leq i\leq 4$*.*

4. Main Results.

In this section, we give the geometric conditions and the main theorems on the extendability of ternary linear codes. For $k\geq 4$, let (C_k-0), (C_k-1) and (C_k-2) be the following conditions:

(C_k-0) there exists a $(\theta_{k-4},0)_{k-3}$ flat δ_1 in Σ^* satisfying $\delta_1\setminus F_0\subset F_e$,

(C_k-1) there is a $(k-2)$-flat Π with $\Pi\setminus F\subset F_e$ containing a $(\theta_{k-4},0)_{k-4}$ flat L such that L is the axis of Π of type $(\theta_{k-4}+3^{k-4},3^{k-4})$.

(C_k-2) there is a $(k-2)$-flat Π with $\Pi\setminus F\subset F_e$ containing a $(\theta_{k-4},0)_{k-4}$ flat L such that L is the axis of Π of type $(\theta_{k-5},2\cdot 3^{k-4})$.

We denote by $\langle \chi_1, \chi_2, \cdots \rangle$ the smallest flat containing subsets $\chi_1, \chi_2, \cdots$ of Σ^*. For $k = 4$ we consider two more conditions:

(C_4-3) there are three non-collinear points $R_1, R_2, R_3 \in F_e$ such that the three lines $\langle R_1, R_2\rangle$, $\langle R_2, R_3\rangle$, $\langle R_3, R_1\rangle$ are $(0,2)$-lines,

(C_4-4) there are three non-collinear points $Q_1, Q_2, Q_3 \in F_1$ such that the three lines $\langle Q_1, Q_2\rangle$, $\langle Q_2, Q_3\rangle$, $\langle Q_3, Q_1\rangle$ are $(0,2)$-lines each of which contains two points of F_e.

For $k \geq 5$, let (C_k-3) and (C_k-4) be the following conditions:

(C_k-3) there is a $(k-2)$-flat Π with $\Pi \setminus F \subset F_e$ containing a $(\theta_{k-5}, 0)_{k-5}$ flat L such that L is the axis of Π of type $(\theta_{k-4}, 2 \cdot 3^{k-4})$.

(C_k-4) there is a $(k-2)$-flat Π with $\Pi \setminus F \subset F_e$ containing a $(\theta_{k-5}, 0)_{k-5}$ flat L such that L is the axis of Π of type $(\theta_{k-4}, 3^{k-4})$.

For $k = 5$ we consider two more conditions:

(C_5-5) there exist a $(4,0)$-line l and four skew $(1,0)$-lines l_1, l_2, l_3, l_4 such that each of $l_1, \ldots, l_4$ meets l and that $\langle l_1, l_2, l_3, l_4\rangle \in \mathcal{F}_3^*$ and $(\cup_{i=1}^4 l_i) \setminus l \subset F_e$ hold,

(C_5-6) there exist a $(2,1)$-line l_0 containing two points $P_1, P_2 \in F_0$ and two $(1,0)$-lines l_1, l_2 (resp. l_1', l_2') through P_1 (resp. P_2) such that $l = \langle l_1, l_2\rangle \cap \langle l_1', l_2'\rangle$ and $m_i = \langle Q_0, Q_i\rangle$ are $(0,2)$-lines for $i = 1, 2$, where $l_0 \cap F_1 = \{Q_0\}$, $l \cap F_1 = \{Q_1, Q_2\}$ and that $(\cup_{i=1}^2 (l_i \cup l_i' \cup m_i)) \setminus F \subset F_e$ holds.

Lemma 4.1 ([11])**.** *Let Δ be a solid in Σ^*.*
(1) *Δ is a $(10,15)$-solid with $\Delta \setminus F \subset F_e$ if and only if Δ satisfies* (C_5-6)*.*
(2) *Δ is a $(16,12)$-solid with $\Delta \setminus F \subset F_e$ if and only if Δ satisfies* (C_5-5)*.*

We define the conditions (C_k-5–C_k-10) for even $k \geq 6$ and $1 \leq s \leq T = (k-4)/2$ as the existence of a $(k-2)$-flat Π with $\Pi \setminus F \in F_e$ satisfying the following conditions, respectively.

(C_k-5) Π contains a $(\theta_{2s-2}, 0)_{2s-2}$ flat which is the axis of Π of type $(\theta_{k-4} - 3^{T+s-1}, \theta_{k-4} + \theta_{T+s-2} + 1)$.

(C_k-6) Π contains a $(\theta_{2s-2}, 0)_{2s-2}$ flat which is the axis of Π of type $(\theta_{k-4} + 3^{T+s-1}, \theta_{k-4} - \theta_{T+s-2})$.

(C_k-7) Π contains a $(\theta_{2s-1}, 0)_{2s-1}$ flat which is the axis of Π of type $(\theta_{k-4}, \theta_{k-4} - \theta_{T+s-1})$.

(C_k-8) Π contains $(\theta_{2s-1}, 0)_{2s-1}$ flat which is the axis of Π of type $(\theta_{k-4}, \theta_{k-4} + \theta_{T+s-1} + 1)$.

(C_k-9) Π contains four $(k-3)$-flats $\pi_1, \cdots, \pi_4$ through a fixed $(\theta_{k-5}, \theta_{k-5} - \theta_T)_{k-4}$ flat Δ such that Δ contains a $(4,0)$-line $l = \{P_1, P_2, P_3, P_4\}$ which is the axis of Δ of type $(\theta_{k-6}, \theta_{k-6} - \theta_{T-1})$ and that P_i is the axis of π_i of type $(\theta_{k-5}, \theta_{k-5} - \theta_{T-1})$.

(C_k-10) Π contains four $(k-3)$-flats $\pi_1, \cdots, \pi_4$ through a fixed $(\theta_{k-5}, \theta_{k-5} + \theta_T + 1)_{k-4}$ flat Δ such that Δ contains a $(4,0)$-line $l = \{P_1, P_2, P_3, P_4\}$ which is the axis of Δ of type $(\theta_{k-6}, \theta_{k-6} + \theta_{T-1} + 1)$ and that P_i is the axis of π_i of type $(\theta_{k-5}, \theta_{k-5} + \theta_{T-1} + 1)$.

We define the conditions (C_k-5) – (C_k-10) for odd $k \geq 7$ and $1 \leq s \leq U+1$ where $U = (k-5)/2$ as the existence of a $(k-2)$-flat Π with $\Pi \setminus F \in F_{\mathrm{e}}$ satisfying the following conditions, respectively.

(C_k-5) Π contains a $(\theta_{2s-1}, 0)_{2s-1}$ flat which is the axis of Π of type $(\theta_{k-4} - 3^{U+s}, \theta_{k-4} + \theta_{U+s-1} + 1)$.

(C_k-6) Π contains a $(\theta_{2s-1}, 0)_{2s-1}$ flat which is the axis of Π of type $(\theta_{k-4} + 3^{U+s}, \theta_{k-4} - \theta_{U+s-1})$.

(C_k-7) Π contains a $(\theta_{2s-2}, 0)_{2s-2}$ flat which is the axis of Π of type $(\theta_{k-4}, \theta_{k-4} - \theta_{U+s-1})$.

(C_k-8) Π contains a $(\theta_{2s-2}, 0)_{2s-2}$ flat which is the axis of Π of type $(\theta_{k-4}, \theta_{k-4} + \theta_{U+s-1} + 1)$.

(C_k-9) Π contains four $(k-3)$-flats $\pi_1, \cdots, \pi_4$ through a fixed $(\theta_{k-5} - 3^{U+1}, \theta_{k-5} + \theta_U + 1)_{k-4}$ flat Δ such that Δ contains a $(4,0)$-line $l = \{P_1, P_2, P_3, P_4\}$ which is the axis of Δ of type $(\theta_{k-6} - 3^U, \theta_{k-6} + \theta_{U-1} + 1)$ and that P_i is the axis of π_i of type $(\theta_{k-5} - 3^U, \theta_{k-5} + \theta_{U-1} + 1)$.

(C_k-10) Π contains four $(k-3)$-flats $\pi_1, \cdots, \pi_4$ through a fixed $(\theta_{k-5} + 3^{U+1}, \theta_{k-5} - \theta_U)_{k-4}$ flat Δ such that Δ contains a $(4,0)$-line $l = \{P_1, P_2, P_3, P_4\}$ which is the axis of Δ of type $(\theta_{k-6} + 3^U, \theta_{k-6} - \theta_{U-1})$ and that P_i is the axis of π_i of type $(\theta_{k-5} + 3^U, \theta_{k-5} - \theta_{U-1})$.

Let C be an $[n,k,d]_3$ code with diversity $(\Phi_0,\Phi_1) \in \mathcal{D}_k^+$, $d \equiv 1$ or 2 (mod 3), $k \geq 3$. Since $\mathcal{D}_3^+ = \{(4,3)\}$, $\mathcal{D}_4^+ = \{(13,9),(10,15),(16,12)\}$ and $\mathcal{D}_k^+ = \Lambda_{k-1}^+$ for $k \geq 5$ ([8], [10]), we have $|\mathcal{D}_k| = 2k-1$ for all $k \geq 3$. It is known that an $[n,4,d]_3$ code with diversity $(\Phi_0,\Phi_1) \in \mathcal{D}_4^+$ is not extendable if $\Phi_e < 3$ for $k=4$ ([8]). The conditions (C_4-0-C_4-4) are used to check the extendability of $[n,4,d]_3$ codes.

Theorem 4.2 ([10])**.** *Let C be an $[n,4,d]_3$ code with diversity $(\Phi_0,\Phi_1) \in \mathcal{D}_4^+$, gcd$(3,d)=1$. Then C is extendable if and only if one of the conditions indicated in Table 3 holds.*

Table 3

(Φ_0,Φ_1)	conditions
(13,9)	(C_4-1), (C_4-4)
(10,15)	(C_4-2), (C_4-3), (C_4-4)
(16,12)	(C_4-0), (C_4-3)

For the case when $k=5$, C is not extendable if $\Phi_e < 9$ when $(\Phi_0,\Phi_1) \neq (40,36)$ or if $\Phi_e < 12$ when $(\Phi_0,\Phi_1) = (40,36)$ ([8]). Otherwise, we need to check whether one of the conditions (C_5-0-C_5-6) holds or not according to the diversity of C.

Theorem 4.3 ([11])**.** *Let C be an $[n,5,d]_3$ code with diversity $(\Phi_0,\Phi_1) \in \mathcal{D}_5^+$, gcd$(3,d)=1$. Then C is extendable if and only if one of the conditions indicated in Table 4 holds.*

Table 4

(Φ_0,Φ_1)	conditions
(40,27)	(C_5-1), (C_5-4)
(31,45)	(C_5-2), (C_5-3), (C_5-4), (C_5-6)
(40,36)	(C_5-4), (C_5-5), (C_5-6)
(40,45)	(C_5-3), (C_5-5), (C_5-6)
(49,36)	(C_5-0), (C_5-3), (C_5-5)

Theorem 4.4 ([9])**.** *Let C be an $[n,k,d]_3$ code with diversity $(\theta_{k-2}, 3^{k-2})$, gcd$(3,d)=1$, $k \geq 6$. Then C is extendable if and only if either the conditions (C_k-1) or (C_k-4) holds.*

Theorem 4.5. *Let C be an $[n,k,d]_3$ code with diversity $(\Phi_0,\Phi_1)\in\mathcal{D}_k^+$, gcd$(3,d)=1$ for $k\geq 6$. Then C is extendable if and only if one of the conditions indicated in Table 5 holds.*

Table 5

(Φ_0,Φ_1) k : even	(Φ_0,Φ_1) k:odd	conditions
$(\theta_{k-2}-3^{T+1},\theta_{k-2}+\theta_T+1)$	$(\theta_{k-2},\theta_{k-2}-\theta_{U+1})$	(C_k-5), (C_k-9), (C_k-10)
$(\theta_{k-2}+3^{T+1},\theta_{k-2}-\theta_T)$	$(\theta_{k-2},\theta_{k-2}+\theta_{U+1}+1)$	(C_k-6), (C_k-9), (C_k-10)
$(\theta_{k-2}-3^{T+s+1},\theta_{k-2}+\theta_{T+s}+1)$	$(\theta_{k-2}-3^{U+s+1},\theta_{k-2}+\theta_{U+s}+1)$	(C_k-5), (C_k-7), (C_k-8)
$(\theta_{k-2}+3^{T+s+1},\theta_{k-2}-\theta_{T+s})$	$(\theta_{k-2}+3^{U+s+1},\theta_{k-2}-\theta_{U+s})$	(C_k-6), (C_k-7), (C_k-8)
$(\theta_{k-2},\theta_{k-2}-\theta_{T+s})$	$(\theta_{k-2},\theta_{k-2}-\theta_{U+s+1})$	(C_k-5), (C_k-6), (C_k-7)
$(\theta_{k-2},\theta_{k-2}+\theta_{T+s}+1)$	$(\theta_{k-2},\theta_{k-2}+\theta_{U+s+1}+1)$	(C_k-5), (C_k-6), (C_k-8)

(T,U and s are defined as in Lemma 2.5)

Proof. We prove this only for $(\Phi_0,\Phi_1)=(\theta_{k-2}-3^{T+1},\theta_{k-2}+\theta_T+1)$ of Theorem 4.5. The others are proved similarly. Let C be an $[n,k,d]_3$ code with diversity $(\theta_{k-2}-3^{T+1},\theta_{k-2}+\theta_T+1)$, $\gcd(3,d)=1$, even $k\geq 6$ where $T=(k-4)/2$.

"only if" part: Assume that C is extendable. Then there is an (i,j)-hyperplane Π satisfying $\Pi\setminus F\subset F_{\rm e}$, where $(i,j)\in\{(\theta_{k-3}-3^{T+1},\theta_{k-3}+\theta_T+1),(\theta_{k-3},\theta_{k-3}-\theta_T),(\theta_{k-3},\theta_{k-3}+\theta_T+1)\}$. If Π is a $(\theta_{k-3}-3^{T+1},\theta_{k-3}+\theta_T+1)_{k-2}$ flat, then (C_k-5) holds with $s=1$ by (2) of Lemma 3.5. If Π is a $(\theta_{k-3},\theta_{k-3}-\theta_T)_{k-2}$ flat, then (C_k-9) holds by (1) of Lemma 3.6. If Π is a $(\theta_{k-3},\theta_{k-3}+\theta_T+1)_{k-2}$ flat, then (C_k-10) holds by (1) of Lemma 3.7.

"if" part: Assume that one of the conditions (C_k-5), (C_k-9), (C_k-10) holds. From the definition of conditions (C_k-5-C_k-10), there exists a $(k-2)$-flat Π with $\Pi\setminus F\in F_{\rm e}$. Hence C is extendable by Lemma 2.3. □

Example. Let C be a $[14,5,7]_3$ code with a generator matrix

$$G=\begin{bmatrix}1&0&0&0&0&0&0&0&1&1&1&1&1&1\\0&1&0&0&0&0&1&1&0&0&1&1&2&2\\0&0&1&0&0&1&0&1&0&1&0&2&1&2\\0&0&0&1&0&1&1&0&1&0&2&2&0&1\\0&0&0&0&1&1&1&2&1&2&0&2&1&0\end{bmatrix},$$

whose weight distribution is

$$0^1 7^{32} 8^{50} 9^{64} 10^{28} 11^{40} 12^{16} 13^{12} \quad (\text{diversity}(40,45),\ \Phi_e = 9).$$

Take four points P, Q_1, Q_2, Q_3 in $\Sigma = \mathrm{PG}(4,3)$ as

$$P = (1,0,0,0,2), Q_1 = (0,1,1,1,0), Q_2 = (0,1,2,2,2), Q_3 = (0,1,2,1,1).$$

Since $wt(P \cdot G) \equiv 0 \pmod 3$, $wt(Q_i \cdot G) \equiv 1 \pmod 3$ and $wt(Q_i \cdot G) \neq 7$, we have $P \in F_0$ and $Q_1, Q_2, Q_3 \in F_e$. One can easily see that $\delta_1 = \langle P, Q_1\rangle$, $\delta_2 = \langle P, Q_2\rangle$, $\delta_3 = \langle P, Q_3\rangle$ are $(1,0)$-lines. It also turns out that the three lines $\langle Q_1, Q_2\rangle$, $\langle Q_1, Q_3\rangle$, $\langle Q_2, Q_3\rangle$ are (0,2)-lines, and that all of the planes not containing P are (4,6)-planes. Thus, P is the axis of type (4,6), that is, (C_5-3) of Theorem 4.3 holds. Hence C is extendable. The solid $\langle P, Q_1, Q_2, Q_3\rangle$ is represented as the variety $V(f)$ with $f = x_0 + 2x_1 + 2x_2 + 2x_3 + x_4$. Hence we can take $h = (1,2,2,2,1)^{\mathrm{T}}$ so that $[G, h]$ generates a $[15,5,8]_3$ code, whose weight distribution is $0^1 8^{60} 9^{40} 10^{62} 11^{20} 12^{40} 13^{10} 14^{10}$.

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