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PARALLEL CLASS INTERSECTION MATRICES OF ORTHOGONAL RESOLUTIONS*

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Abstract. Parallel class intersection matrices (PCIMs) have been defined and used in [6], [14], [15] for the classification of resolvable designs with several parameter sets. Resolutions which have orthogonal resolutions (RORs) have been classified in [19] for designs with some small parameters. The present paper deals with the additional restrictions that the existence of an orthogonal mate might impose on the PCIMs of a resolution, and with the effect of both PCIMs usage and the methods for RORs construction described in [19] and [20]. It is shown in several examples how consideration of PCIMs can result in constructing only of solutions which can have orthogonal mates, and thus substantially improve the computation time. There are parameters for which PCIMs make the classification of RORs possible, and also cases when PCIMs directly prove the nonexistence of doubly resolvable designs with certain parameters.

1. Introduction. For the basic concepts and notations concerning combinatorial designs and their resolvability refer, for instance, to [2], [3], [21].

Let $V = \{P_i\}_{i=1}^{v}$ be a finite set of *points*, and $\mathcal{B} = \{B_j\}_{j=1}^{b}$ a finite collection of k-element subsets of V, called *blocks*. If any 2-subset of V is contained

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in exactly λ blocks of $\mathcal{B}$, then $D = (V, \mathcal{B})$ is a 2-(v, k, λ) *design*, or *balanced incomplete block design (BIBD)*.

Two designs are *isomorphic* if there exists a one-to-one correspondence between the point and block sets of the first design and respectively, the point and block sets of the second design, and if this one-to-one correspondence does not change the incidence. An *automorphism* is an isomorphism of the design to itself, i.e. a permutation of the points that transforms the blocks into blocks.

A *resolution* is a partition of the blocks into subsets called *parallel classes* such that each point is in exactly one block of each parallel class. A parallel class contains v/k blocks and a resolution $\mathcal{R}$ consists of $r = (b.k/v)$ parallel classes, $\mathcal{R} = \mathcal{R}_1, \ldots, \mathcal{R}_r$. The design is *resolvable* (RBIBD) if it has at least one resolution.

Two resolutions are isomorphic if there exists an automorphism of the design transforming each parallel class of the first resolution into a parallel class of the second one.

Let $Z_q = \{0, 1, \ldots, q-1\}$. A word of length r over Z_q is an r-tuple $x = (x_1, x_2, \ldots, x_r) \in Z_q^r$. The Hamming distance $d(x, y)$ between two words $x, y \in Z_q^r$ is the number of coordinates in which the words differ. An equidistant $(r, v, d)_q$ code is a set of v words of length r over Z_q, with the property that the distance between any two distinct words is d. There is a one-to-one correspondence [18] between the resolutions of 2-(qk, k, λ) designs and the $(r, qk, r-\lambda)_q$ equidistant codes with maximal minimum distance.

Consider two resolutions $\mathcal{R}$ and $\mathcal{T}$ of one and the same design. A parallel class T_i $(i = 1, 2, \ldots, r)$ of $\mathcal{T}$ is *orthogonal* to $\mathcal{R}$ if the number of blocks in $T_i \cap R_j$ is either 0 or 1 for all $1 \leq j \leq r$. The resolutions $\mathcal{R}$ and $\mathcal{T}$ are orthogonal if all classes of $\mathcal{T}$ are orthogonal to $\mathcal{R}$. Orthogonal resolutions may or may not be isomorphic to each other. A *doubly resolvable design (DRD)* is a design which has at least two orthogonal resolutions. We denote by *ROR* a resolution which is orthogonal to at least one other resolution.

An affine resolvable design is an RBIBD with the property that any two blocks from different parallel classes are incident to exactly k^2/v common points.

Resolvability and double resolvability are important properties of a design for many practical applications such as the design and analysis of experiments and some schemes in cryptography.

Papers on DRDs mainly deal with setting of the existence problem (see for instance [1], [4], [10], [11], [12], [17], [22]). Examples of using parallel classes intersection matrices (PCIMs) for the construction of resolvable designs can be found for instance in [6], [14] and [15]. In [14] and [15] they are used to produce initial structures of the resolution of the designs and in [6] for partial verification of the classification of resolvable 2-$(14, 7, 12)$ designs.

There are many papers on the existence or classification of resolvable 2-(v,k,λ) designs with given parameters, see for instance [5], [7], [8], [16]. A very good recent survey of the different approaches for constructing and classifying design resolutions is contained in [9]. The methods for construction of RORs presented in [19] and [20] use the word by word orderly generation described in [9], but from some word on an orthogonal resolution existence (ORE) test [20] is applied. This makes the computation faster, and full classification of RORs of designs with some small parameters is possible.

We will show below that if not all resolutions, but only RORs are constructed, additional restrictions can be set on the intersection between parallel classes and they can make the computation faster. In the present work this approach is used to remove some of the constructions which cannot give RORs. It is especially effective for design resolutions, which have 2 blocks in the parallel class.

2. Intersection between two parallel classes – PCIM patterns. We will use the terminology developed in [6], [14], [15]. Let $q = v/k$ be the number of the blocks in a parallel class. Consider two parallel classes $R_x = \{R_{x1}, R_{x2}, \ldots, R_{xq}\}$ and $R_y = \{R_{y1}, R_{y2}, \ldots, R_{y_q}\}$ of a resolution $\mathcal{R}$, $x, y = 1, 2, \ldots, r$. Define their *parallel classes intersection matrix (PCIM)* as $P_{x,y} = (p_{i,j})_{q\times q}$, where $p_{i,j} = |R_{xi} \cap R_{y_j}|$ for $i, j = 1, 2, \ldots, q$. Each point is in exactly one block of a parallel class, so $p_{i,1} + p_{i,2} + \ldots + p_{i,q} = k$ and $p_{1,j} + p_{2,j} + \ldots + p_{q,j} = k$ for any $i, j = 1, 2, \ldots, q$.

Without loss of generality we fix $R_1 = \{\{1, 2, \ldots, k\}, \{k+1, k+2, \ldots, 2k\}, \ldots, \{v-k+1, v-k+2, \ldots, v\}\}$ and by computer search we produce all inequivalent intersection patterns $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_u$ for the PCIM $P_{1,y}$ (note that the possibilities for $P_{1,y}$ are the same for any $y = 2, 3, 4, \ldots, r$).

Example 1. Designs with parameters 2-$(9,3,2)$ have 9 nonisomorphic resolutions, here we present the incidence matrix of one of them. The design has 24 blocks partitioned in 8 parallel classes.

$$\left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc|ccc|ccc}
1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0\\
1&0&0&1&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0\\
1&0&0&1&0&0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1&0&0&1\\
0&1&0&0&1&0&1&0&0&1&0&0&0&1&0&0&1&0&0&0&1&0&0&1\\
0&1&0&0&1&0&0&1&0&0&1&0&0&0&1&0&0&1&1&0&0&1&0&0\\
0&1&0&0&0&1&1&0&0&0&0&1&1&0&0&0&0&1&0&1&0&0&1&0\\
0&0&1&0&1&0&0&0&1&1&0&0&0&0&1&1&0&0&0&1&0&0&1&0\\
0&0&1&0&0&1&0&1&0&0&0&1&0&1&0&1&0&0&1&0&0&0&0&1\\
0&0&1&0&0&1&0&0&1&0&1&0&1&0&0&0&1&0&0&0&1&1&0&0
\end{array}\right)$$

The possibilities for the intersection patterns in this example are:

$$\mathcal{P}_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \mathcal{P}_2 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad \mathcal{P}_3 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\mathcal{P}_4 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \quad \mathcal{P}_5 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and for the resolution above $P_{1,2}$ is of type $\mathcal{P}_2$, $P_{1,3}$ and $P_{1,6}$ are of type $\mathcal{P}_3$ and $P_{1,4}$, $P_{1,5}$, $P_{1,7}$ and $P_{1,8}$ are of type $\mathcal{P}_5$.

Let $z_n = |\{R_x : P_{1,x} = \mathcal{P}_n, x = 1, 2, \ldots, r\}|$, $n = 1, 2, \ldots, u$. Since the first parallel class meets other classes in $r - 1$ PCIMs,

$$\sum_{i=1}^{u} z_i = r - 1. \tag{1}$$

Denote by ψ_n the number of pairs of points incident both with the blocks of the first parallel class and with the blocks of a parallel class R_y such that $P_{1,y} = \mathcal{P}_n$. Then

$$\sum_{n=1}^{u} \psi_n . z_n = v(k-1)(\lambda-1)/2 \tag{2}$$

The solutions of these equations give the possible intersection patterns of the parallel classes of the resolutions of the design.

In Example 1 equations (1) and (2) give

$$\begin{gathered} z_1 + z_2 + z_3 + z_4 + z_5 = 7 \\ 9.z_1 + 5.z_2 + 2.z_3 + 3.z_4 + 0.z_5 = 9 \end{gathered}$$

with the following 4 nonnegative integer solutions:

z_1	1	0	0	0
z_2	0	0	1	0
z_3	0	0	2	3
z_4	0	3	0	1
z_5	6	4	4	3

This means that a resolution of a 2-$(9, 3, 2)$ design has one of these 4 intersection patterns. The resolution which we consider corresponds to the third solution.

3. Double resolvability restrictions on PCIMs and the classification of RORs. The ROR construction methods from [19] imply construction of the corresponding to the resolution equidistant code word by word, using orderly generation to which an orthogonal resolution existence (ORE) test [20] is added from some word on. The construction only of resolutions with certain intersection patterns is easy to implement with these methods.

A DRD has at least two orthogonal resolutions, i.e. two parallel classes, one of the first and one of the second resolution, have at most one common block. We check if the obtained intersection patterns allow a combination of blocks of different parallel classes in a parallel class orthogonal to the resolution . For some parameters it is possible to reject intersection patterns which cannot yield a ROR, and this makes the computation much faster.

If $q = 2$, we have two blocks in a parallel class and if we already fix the first parallel class as described above, a ROR has at least one parallel class incident with the same points as the first one, so only solutions with $z_1 > 0$, where $\mathcal{P}_1 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ can produce a ROR.

For some designs all intersection patterns might be rejected, and the nonexistence of RORs with these parameters is proved without any further attempt to construct a resolution. For instance if we consider 2-$(14, 7, 12)$ designs, there are two blocks in the parallel class and so the PCIMs [6] are:

$$\mathcal{P}_1 = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}, \quad \mathcal{P}_2 = \begin{pmatrix} 6 & 1 \\ 1 & 6 \end{pmatrix}, \quad \mathcal{P}_3 = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}, \mathcal{P}_4 = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$$

As a result of equations (1) and (2) there are two nonnegative integer solutions, and in both of them $z_1 = 0$:

z_1	0	0
z_2	1	0
z_3	0	3
z_4	24	22

That is why such solutions cannot give a DRD. So we can conclude that a doubly resolvable design with parameters 2-$(14, 7, 12)$ does not exist.

Designs with parameters 2-$(2k, k, k-1)$ are affine. In this case any two blocks from different parallel classes intersect in a constant number of points. For these parameters it is half of the block size and therefore only one type of PCIM is possible $\mathcal{P}_y = \begin{pmatrix} k/2 & k/2 \\ k/2 & k/2 \end{pmatrix}$, $y = 2, 3, \ldots, r$. So we can conclude that designs with parameters 2-$(2k, k, k-1)$ can not be doubly resolvable.

The solutions of equations (1) and (2) and the restrictions on PCIMs imposed by double resolvability give only a necessary condition for the possible existence of orthogonal resolutions.

For 2-$(12,6,10)$ designs $k=6$ and the PCIMs are:

$$\mathcal{P}_1 = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \quad \mathcal{P}_2 = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}, \quad \mathcal{P}_3 = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}, \mathcal{P}_4 = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

In this case, it holds that

$$z_1 + z_2 + z_3 + z_4 = 21$$
$$30.z_1 + 20.z_2 + 14.z_3 + 12.z_4 = 270$$

with 3 nonnegative integer solutions

z_1	1	0	0
z_2	0	2	1
z_3	0	1	5
z_4	20	18	15

We investigate the first one as possibly doubly resolvable and find 1 resolution of 1 doubly resolvable design.

For 2-$(12,6,15)$ designs the PCIMs are the same as for 2-$(12,6,10)$ designs and in this case, it holds:

$$z_1 + z_2 + z_3 + z_4 = 32$$
$$30.z_1 + 20.z_2 + 14.z_3 + 12.z_4 = 420$$

with 9 nonnegative integer solutions

z_1	2	1	1	1	0	0	0	0	0
z_2	0	2	1	0	4	3	2	1	0
z_3	0	1	5	9	2	6	10	14	18
z_4	30	28	25	22	26	23	20	17	14

We investigate the first four of them as possibly doubly resolvable. We obtain positive result for RORs only from the first PCIM pattern – 1 resolution of 1 doubly resolvable design. We also find altogether 225970 nonisomorphic resolutions of resolvable designs (the previous bound was ≥ 11604 [13]).

For 2-$(16,8,7)$ designs $k=8$, and the PCIMs are:

$$\mathcal{P}_1 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}, \quad \mathcal{P}_2 = \begin{pmatrix} 7 & 1 \\ 1 & 7 \end{pmatrix}, \quad \mathcal{P}_3 = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix},$$

$$\mathcal{P}_4 = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}, \quad \mathcal{P}_5 = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$$

In this case equations (1) and (2) give:

$$z_1 + z_2 + z_3 + z_4 + z_5 = 29$$
$$56.z_1 + 42.z_2 + 32.z_3 + 26.z_4 + 24.z_5 = 728$$

with the following 8 nonnegative integer solutions:

z_1	1	0	0	0	0	0	0	0
z_2	0	1	0	1	0	0	0	0
z_3	0	0	0	1	1	3	2	4
z_4	0	7	16	3	12	4	8	0
z_5	28	21	13	24	16	22	19	25

Only from the first one we can get doubly resolvable designs. By computer search 5 resolutions of doubly resolvable designs and 1895 resolutions of resolvable designs are found.

For 2-$(18, 9, 16)$ designs $k = 9$ and the PCIMs are:

$$\mathcal{P}_1 = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}, \quad \mathcal{P}_2 = \begin{pmatrix} 8 & 1 \\ 1 & 8 \end{pmatrix}, \quad \mathcal{P}_3 = \begin{pmatrix} 7 & 2 \\ 2 & 7 \end{pmatrix},$$

$$\mathcal{P}_4 = \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}, \quad \mathcal{P}_5 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

In this case, it holds that

$$z_1 + z_2 + z_3 + z_4 + z_5 = 33$$
$$72.z_1 + 56.z_2 + 44.z_3 + 36.z_4 + 32.z_5 = 1080$$

with 4 nonnegative integer solutions:

z_1	0	0	0	0
z_2	0	0	0	1
z_3	0	1	2	0
z_4	6	3	0	0
z_5	27	29	31	32

None of these solutions can lead to a ROR, therefore the nonexistence of 2-$(18, 9, 16)$ DRDs is proved.

For 2-$(20, 10, 18)$ designs $k = 10$, and the PCIMs are:

$$\mathcal{P}_1 = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}, \quad \mathcal{P}_2 = \begin{pmatrix} 9 & 1 \\ 1 & 9 \end{pmatrix}, \quad \mathcal{P}_3 = \begin{pmatrix} 8 & 2 \\ 2 & 8 \end{pmatrix}, \quad \mathcal{P}_4 = \begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix},$$

$$\mathcal{P}_5 = \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}, \quad \mathcal{P}_6 = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$$

In this case, it holds that

$$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 37$$
$$90.z_1 + 72.z_2 + 58.z_3 + 48.z_4 + 42.z_5 + 40.z_6 = 1530$$

with 19 nonnegative integer solutions, one of them can lead to ROR:

z_1	z_2	z_3	z_4	z_5	z_6
1	0	0	0	0	36

Three resolutions of three doubly resolvable designs are obtained.

These results are summarized in the next table. The first column shows the number of the design in the table of [13], the second its parameters. In the third column the number of nonisomorphic resolutions Nr is presented. It is followed by $*$ if the value is not obtained by the methods discussed above, but is taken from [13]. In the column ROR the number of nonisomorphic resolutions with orthogonal resolution is given and in the last column the doubly resolvable designs with these parameters. All results for RORs except the result for 2-$(20, 10, 18)$ have also been obtained by fixing only the first two symbols in each word (the second symbol equal to the first one) and without restrictions imposed by the PCIMs on the rest of the structure of the resolution. PCIMs usage makes the classification of the RORs of 2-$(20, 10, 18)$ designs possible, while in the other cases it substantially reduces the computation time.

No	BIBD	Nr	ROR	DRD
319	(12,6,10)	545*	1	1
451	(14,7,12)	1363486*	0	0
618	(16,8,14)	$\geq$ 1895	5	5
743	(12,6,15)	$\geq$ 225970	1	1
791	(18,9,16)	$\geq$ 1*	0	0
1007	(20,10,18)	$\geq$ 4*	3	3

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