ON THE VERTEX SEPARATION OF MAXIMAL OUTERPLANAR GRAPHS

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ABSTRACT. We investigate the NP-complete problem VERTEX SEPARATION (VS) on Maximal Outerplanar Graphs (mops). We formulate and prove a "main theorem for mops", a necessary and sufficient condition for the vertex separation of a mop being k. The main theorem reduces the vertex separation of mops to a special kind of stretchability, one that we call affixability, of submops.

1. Introduction. VERTEX SEPARATION is an NP-complete problem on undirected graphs with numerous practical applications in diverse areas such as natural language processing, VLSI design, network reliability, computational biology and others. It is equivalent to other, seemingly unrelated, graph problems such as NODE SEARCH NUMBER, INTERVAL THICKNESS and the famous PATHWIDTH. For a brief survey of some theoretical and practical applications of VERTEX SEPARATION and its relation to other problems, see [10].

The NP-completeness of a computational problem is a strong evidence that it is intractable in general. There are polynomial time algorithms for VS on certain classes of graphs. A very important class of graphs for which there exist polynomial time algorithms are graphs whose treewidth is bound by a constant k, also called partial k-trees. However, those algorithms have only theoretical importance because the polynomial upper bound on the running time has a high degree, *e.g.* the algorithm of Bodlaender and Kloks on graphs of treewidth k runs in time $\Omega(n^{4k+3})$ ([2]), and because they are extremely difficult to implement in practice. The ongoing construction of approximate algorithms for the VS of outerplanar graphs (see [8], [1], [3]) that have treewidth only 2 is an evidence for the fact that the precise algorithms at the moment are inacceptably slow.

In the context of partial k-trees for fixed k, till recently the only fast, practical algorithm was the linear-time algorithm for VERTEX SEPARATION on trees, *i.e.* 1-trees, due to Ellis, Sudborough, and Turner [7]. Mops are, in a sense, binary 2-trees (we explain later precisely what we mean). A fast and practical algorithm for VERTEX SEPARATION on mops, in our opinion, would be an important first step towards a fast and practical algorithm for VERTEX SEPARATION on 2-trees in general.

2. Background.
2.1. Basic definitions. We assume the reader is familiar with the basic definitions of Graph Theory. We consider undirected graphs without multiple edges or loops. If G is a graph and we write "$u \in G$", we mean that u is a vertex in G. By $V(G)$ we denote the vertex set of a graph G and by $E(G)$, the edge set. For any vertex $u \in G$, by "adj(u)" we denote the set of vertices that u is adjacent to. By "path" we mean a simple path and by "cycle" we mean a simple cycle. The length of a path p is the number of vertices in it and is denoted by $|p|$. If $p = u_1, u_2, \ldots, u_k$ is a path, we say that u_1 and u_k are *the endpoints of* p.

To *delete* a vertex u from $G = (V, E)$ means to tranform G into $G_1 = (V \setminus \{u\}, E \setminus E_u)$, where E_u is the set of edges in G with one endpoint u. We write $G_1 = G - u$. We generalise that operation to deletion of a vertex set $U \subset V$, writing $G_1 = G - U$. To *remove* an edge e from $G = (V, E)$ means to tranform G into $(V, E \setminus \{e\})$. To delete a subgraph G' from G means to delete all the vertices of G' from G.

A graph G is *connected* if there is a path between any two of its vertices. Suppose that G is connected and $V(G)$ is partitioned into three (nonempty) subsets V_1, V_2, and V_3 such that for any $u \in V_1$ and any $v \in V_3$, any path with endpoints u and v has at least one vertex from V_2. Then we say that V_2 is *a*

separator of G and it *separates* V_1 *and* V_3. If $V_2 = \{w\}$, then w is *a cut vertex.* Let k be a positive integer. *A* k-*clique* is a complete graph with k vertices. G is k-*connected* if either G is a $(k+1)$-clique, or $|V(G)| \geq k+2$ and G does not have a separator of less than k vertices. A connected graph having more than one vertex and no cut vertex is called *a block*. If G is a connected graph, *a block of* G is a maximal induced subgraph of G that is a block.

Let $G = (V, E)$ be a graph, and $U \subset V$ be a minimal separator of G. Let $G - U$ have t connected components $G_1, G_2, \ldots, G_t$ for some $t \geq 2$. For $1 \leq i \leq t$, let G^i be the subgraph of G, induced by $V(G_i) \cup U$. We call G^1, G^2, $\ldots$, G^t, *the components of* G *after splitting* U. The operation that is opposite of splitting is called *gluing*: if there are two or more distinct graphs whose vertices have distinct names with the exception that each graph has t vertices, for some $t \geq 1$, having the same names—say $u_1, u_2, \ldots, u_t$—in all those graphs, to *glue* these graphs together means to built a single graph out of them by identifying every u_i in every graph.

2.2. k-trees.

Definition 1 (k-tree)**.** *Let* k *be a positive integer.* k-*trees are these and only these graphs that are constructed according to the following rules:*

1. *A* $(k+1)$-*clique is a* k-*tree.*

2. *If* $G = (V, E)$ *is a* k-*tree,* H *is a* k-*clique subgraph of* G, *and* u *is a vertex not in* G, *then* $G' = (V \cup \{u\}, E \cup E')$ *is a* k-*tree, where* $E' = \{(u, w) \,|\, w \in V(H)\}$.

Some authors, *e.g.* [5], consider the k-clique to be a k-tree. We find it more convenient to abide by the alternative convention, *e.g.* [4] (Definition 6.43, pp. 99) and [9] (Definition 2.1.8, pp. 12), in which a k-tree has at least $k+1$ vertices. Very informally speaking, the k-tree consists of $(k+1)$-cliques, glued together in a "tree-like manner".

If it is the case that any k-clique in a k-tree is a subgraph of at most two $(k+1)$-cliques, we call the k-tree, *simple*. Informally, simple k-trees are those k-trees in which any k-clique H refered to in Definition 1, part 2, is never used again for adding vertices to the k-tree. The distinction between a 2-tree that is simple and one that is not simple is shown on Figure 1 on page 212.

2.3. Mops. A graph G is *planar* if it can be drawn in the Euclidean plane so that each vertex corresponds to a distinct point called *plane vertex*, each edge corresponds to a simple open curve called *a plane edge*, and each plane edge

connects the two plane vertices that correspond to the endvertices of the original graph edge. Further, no two plane edges intersect, except at common endpoints. Suppose that G is drawn in this way. Then the plane plus the drawing in it is called *a planar embedding of* G.

Let $G = (V, E)$ be a planar graph, and $\mathcal{G}$ be a planar embedding of G. Remove from the plane all the plane vertices and plane edges. The remainder of the plane falls into connected components, called the *faces* of the embedding. The *boundary* of a face is the set of plane edges in its closure. There is a face that is not bounded called *the outer face*. All the other faces, if such exist, are *the inner faces*. For some vertex u and some face $\mathcal{F}$, we say that u *belongs to* $\mathcal{F}$, denoted by $u \in \mathcal{F}$, if the plane vertex corresponding to u is an endpoint in at least one of the bounding plane edges of $\mathcal{F}$. *The dual planar graph of* G *with respect to* $\mathcal{G}$, or simply *the dual of* G, is the graph $D = (V_D, E_D)$, such that:

- $V_D \cap V = \emptyset$,

- each vertex from V_D corresponds to a unique face of $\mathcal{G}$,

- $\forall x, y \in V_D$, $(x, y) \in E_D$ if and only if the faces corresponding to x and y share one or more edges.

Clearly, D is a planar graph, too. Our definition of dual differs from the commonly accepted one which says there is an edge between any two vertices of the dual graph for each edge of the original graph that is shared between the corresponding faces of the embedding. It is easy to see that according to the common definition, the dual graph has, in general, multiple edges between its vertices, something that we would like to avoid. It is a well-known fact that in general the dual graph is not uniquely determined by the original planar graph, *i.e.* for diffent planar embeddings of G, the duals may not be isomorphic. That facts holds for our definition of dual, too. It is easy to see that for the particular class of graphs we consider, *viz.* mops, their weak duals (see the definitions below) are uniquely determined, so we talk about the weak dual of a mop without specifying any embedding.

Let $\mathcal{P}$ be a property defined on graphs. We say that a graph $G = (V, E)$ is *edge-maximal with respect to* $\mathcal{P}$ in case that $\mathcal{P}$ holds for G but for any $u, v \in V$ such that $(u, v) \notin E$, $\mathcal{P}$ does not hold for $G' = (V, E \cup \{(u, v)\})$.

Definition 2 (mop). *A planar graph* G *is* outerplanar *if there exists a plane embedding* $\mathcal{G}$ *of* G, *such that all the vertices of* $\mathcal{G}$ *belong to the outer face. An outerplanar graph* G *is* edge-maximal outerplanar, *or simply* maximal

outerplanar*, if* G *is edge-maximal with respect to planarity. We call a maximal outerplanar graph,* a mop*. If* G *is a mop, we call* a submop *any subgraph* G₁ *of* G*, such that* G₁ *is a mop.*

The name "mop" comes from **m**aximal **o**uter**p**lanar. A single edge (u, v) is considered to be a *trivial* mop. Any mop with more than two vertices is called *nontrivial.* It is easy to see that any mop with four or more vertices is 2-connected. When we say "face of a mop" we mean an inner face. Note that each face of any planar embedding of any mop has three bounding planar edges, for if there were four or more bounding edges, the graph would not be maximal outerplanar. We blur the distinction between a graph and its planar embedding; thus, in the context of mops, we use the term "face" as a synonym for "3-clique". *The weak dual* of any outerplanar graph is the dual with the vertex corresponding to the outer face deleted. It is easy to see that the weak dual of an outerplanar graph is a forest: if there was a cycle in the weak dual, there would be a vertex in the original graph that is not on the outer face. Clearly, the weak dual of any nontrivial mop is a tree. Figure 1(a)shows a mop and its weak dual.

Lemma 1. *Nontrivial mops are precisely the simple* 2-*trees.*

P r o o f. Let G be a nontrivial mop. We show G is a simple 2-tree. Let G' be the weak dual of G. As already observed, G' is a tree. Choose any 3-clique C in G and let a be the vertex of G' corresponding to C. The key observation is that G can be built from C by performing a breadth-first search in G' starting with a – every time we visit a hitherto unvisited vertex x of G', we glue the clique that corresponds to it to the hitherto built G.

Let G be a simple 2-tree. Then G can be obtained from a 3-clique using the construction of Definition 1. It is trivial to show that G is outerplanar, having in mind that once an edge has been used for adding a vertex to it, it is never used again for that purpose – if we think of the planar embedding of the graph in construction, the planar vertex that corresponds to the newly added vertex can always be placed within the outer face, and so all the vertices remain on the outer face. The fact that the constructed graph is maximal outerplanar is obvious. □

Figure 1 illustrates the distinction between simple 2-trees, *i.e* mops, and 2-trees in general: G_2 is not simple because the egde (u, v) belongs to three faces.

The edges of the boundary of the outer face are *the outer edges.* Any edge that is not outer is *inner edge.* Clearly, the trivial mop and the 3-clique have only outer edges and so the smallest mop with an inner edge is the four vertex mop. The outer edges of a mop form a cycle. We call this cycle *the periphery.* The two vertex set of each inner edge is a separator for the mop. When we say that

(a): A mop (a simple 2-tree) and its weak dual drawn inside with wide gray lines.

(b): A 2-tree that is not a mop − the 2-clique induced by $\{u, v\}$ belongs to three 3-cliques.
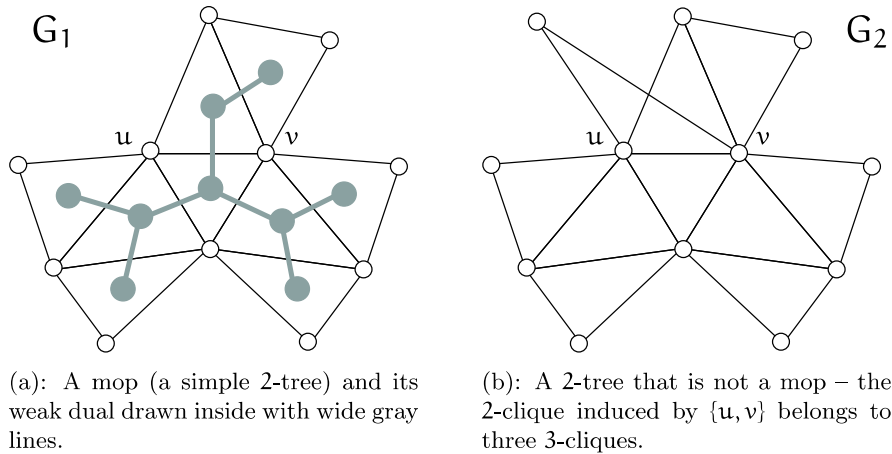
Fig. 1. Mops are a strict subclass of 2-trees

we *split an inner edge*, we mean that we split the set of the vertices of that edge. Note that there are precisely two nontrivial mops after splitting any inner edge.

Definition 3. *A nontrivial mop is a* a 2-path *if its weak dual is a path.*

Let $\mathcal{P}$ be a 2-path with at least two faces. The faces of $\mathcal{P}$ that correspond to the leaves of the weak dual are called *the endfaces of* $\mathcal{P}$, and we say that $\mathcal{P}$ *2-connects* them. See Figure 2 for illustration. Sometimes we describe a 2-path by enumerating its faces as we do on Figure 2.
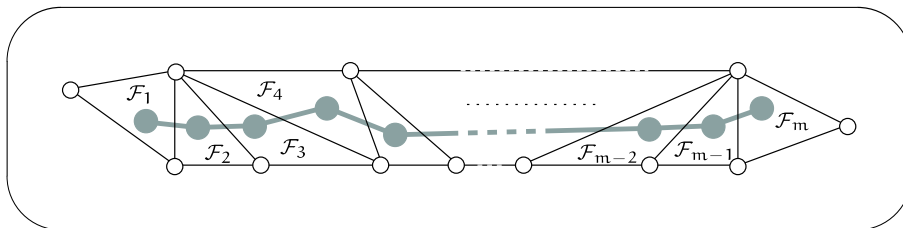

Fig. 2. A 2-path $\mathcal{P} = \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$. $\mathcal{F}_1$ and $\mathcal{F}_m$ are the endfaces and $\mathcal{P}$ 2-connects $\mathcal{F}_1$ and $\mathcal{F}_m$. The weak dual is drawn with thick gray lines

It is easy to show that in any mop G, there is a unique 2-path that 2-connects any two faces. Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be any two edges in a mop that may or may not share an endpoint. Define $\mathcal{F}^i$ as follows for $i = 1, 2$. If e_i is an outer edge, $\mathcal{F}^i$ is the (unique) face that e_i belongs to. Otherwise, e_i belongs to two faces, say $\mathcal{F}'$ and $\mathcal{F}''$. Let G' and G'' be the two submops

after splitting e_i so that G' contains $\mathcal{F}'$ and G'' contains $\mathcal{F}''$. Then $\mathcal{F}^i = \mathcal{F}'$ if $e_{(i \bmod 2)+1}$ is in G', else $\mathcal{F}^i = \mathcal{F}''$. *The 2-path connecting e_1 and e_2* is the 2-path connecting $\mathcal{F}^1$ and $\mathcal{F}^2$.

2.4. Submops begotten by an edge and by a face. The following definitions are illustrated on Figure 3. Let $G = (V, E)$ be a mop and let $\mathcal{F} = (u, v, w)$ be a face in it (see Figure 3(a)). For any edge e among the edges (u, v), (u, w), and (v, w), if e is an outer edge then *the submop begotten by $\mathcal{F}$ relative to e* is e itself. For instance, on Figure 3(a) the submop begotten by $\mathcal{F}$ relative to (u, w) is (u, w).

Otherwise, *i.e.* if e is an inner edge, the submop begotten by $\mathcal{F}$ relative to e is the submop G_1 of G that is the connected component after splitting e that does not contain $\mathcal{F}$ (see Figure 3(b)). The other component, *viz.* the one containing $\mathcal{F}$, is denoted by $G \ominus G_1$ (see Figure 3(c)). Collectively, G_1, G_2, and G_3 are called *the submops begotten by $\mathcal{F}$* (see Figure 3(b)).
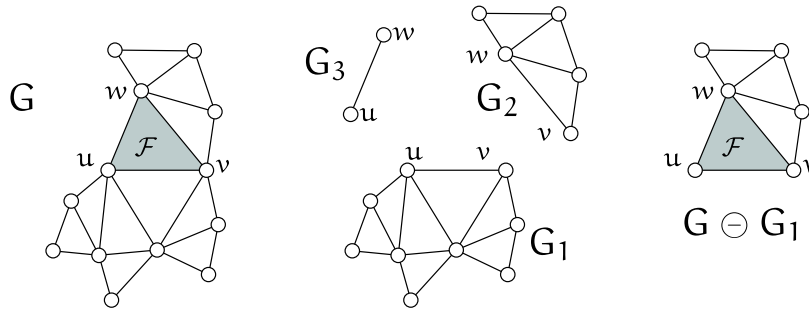

Fig. 3. Illustrating the idea of begotten submops

2.5. Rooted mops. Let $G = (V, E)$ be a non-trivial mop and $\mathcal{F} = (u, v, w)$ be a face in it, such that the edge (u, v) is an outer edge. We make G *a rooted mop* with *root edge* (u, v) by postulating (u, v) is special, distinct from all the other edges. The face $\mathcal{F}$ is *the root face*. The submops begotten by $\mathcal{F}$ relative to the edges (v, w) and (u, w) are *the children of $\mathcal{F}$*. The two children are rooted mops themselves, with root edges (u, w) and (v, w), respectively. Note that in a rooted mop each face has two children. We consider the trivial mop to be rooted, although it can have no root face. Let (x, y) be any inner edge in G. By "$G[x, y]$" we denote the submop rooted at (x, y) (see Figure 4).

The mop $G[x, y]$ can be specified in terms of its the root face rather than in terms of x and y, in case it is nontrivial. Suppose that $\mathcal{F}_1$ is the root face of
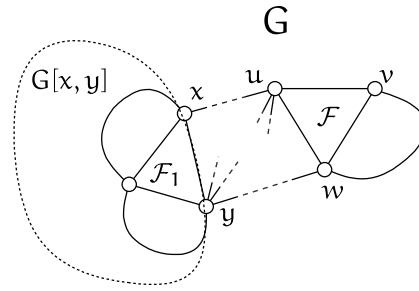
Fig. 4.

$G[x, y]$, as shown on Figure 4. Then "$G[\mathcal{F}_1]$" is another name for $G[x, y]$. We emphasise the notations "$G[x, y]$" and "$G[\mathcal{F}_1]$" are used only in the context of rooted mops.

Definition 4. *Suppose G is a mop rooted at some edge (u, v). Suppose that (x, y) is an inner edge not necessarily vertex disjoint with (u, v). Clearly, in G there are exactly two vertex disjoint paths p', p'', each having one endpoint in $\{u, v\}$ and the other endpoint in $\{x, y\}$, such that each of them has only outer edges and contains precisely one vertex from $\{u, v\}$ and precisely one vertex from $\{x, y\}$. Precisely one of the following is the case:*

- *one of p' and p'' connects u with x and the other one connects v with y, in which case we say that u is peripherally closer to x and v is peripherally closer to y;*

- *one of p' and p'' connects u with y and the other one connects v with x, in which case we say that u is peripherally closer to y and v is peripherally closer to x.*

Collectively, p' and p'' are the peripheral paths with respect to x and y, *or simply* the peripheral paths *in case that x and y are understood.*

For instance, on Figure 4, u is peripherally closer to x and v is peripherally closer to y. We emphasise that being peripherally closer has nothing to do with the shortest path in the graph: it is easy to imagine that in G from Figure 4, the shortest path from y to u is shorter than the peripheral path from y to v, and yet we say that y is peripherally closer to v. Note that $\{u, v\}$ and $\{x, y\}$ are not necessarily disjoint though they cannot be equal if (x, y) is an inner edge. So if, say, $x = u$, by Definition 4 we have that x is peripherally closer to u and y is peripherally closer to v – just as one might expect intuitively.

2.6. The children of a 2-path. Severing a 2-path. Let G be a non-rooted mop and $\mathcal{P} = \mathcal{F}_1 \; \mathcal{F}_2 \; \cdots \; \mathcal{F}_m$, for some $m \geq 2$, be a 2-path in it. Consider $\mathcal{P}$ as a mop and let e_1' and e_1'' be the two peripheral edges of $\mathcal{F}_1$, and let e_m' and e_m'' be the two peripheral edges of $\mathcal{F}_m$, within $\mathcal{P}$. Consider the following $m + 2$ submops of G:

- the two submops begotten by $\mathcal{F}_1$ relative to e_1' and e_1'',

- the two submops begotten by $\mathcal{F}_m$ relative to e_m' and e_m'',

- the $m - 2$ submops begotten by each of $\mathcal{F}_2, \ldots, \mathcal{F}_{m-1}$, relative to its (only) peripheral edge.

Collectively, those submops are known as the *children of* $\mathcal{P}$. Each child is considered to be a rooted mop, rooted at the respective peripheral edge of $\mathcal{P}$. See Figure 5.
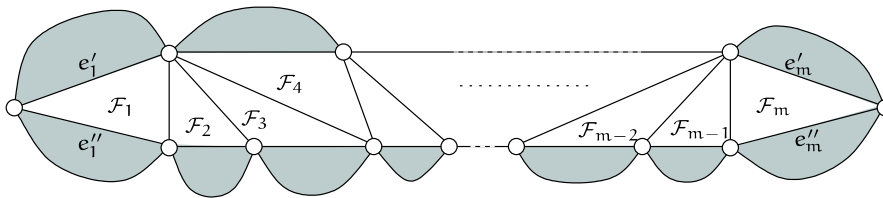

Fig. 5. $\mathcal{P} = \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ and the children of $\mathcal{P}$ are drawn in gray

If $\mathcal{P}$ has a single face, the children of $\mathcal{P}$ are the three submops begotten by $\mathcal{P}$ but being considered as rooted mops.

We define two special types of 2-paths: *terminated* and *semi-terminated*. First assume that $\mathcal{P} = \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ is a 2-path with at least two faces. We choose one child of $\mathcal{P}$ rooted at an edge of $\mathcal{F}_1$ and one child relative to an edge of $\mathcal{F}_m$ and define them to be *terminators*. Once defined as terminators, they are not considered as children of $\mathcal{P}$ any more. We emphasise that the choice of terminators is not unique – at each end of $\mathcal{P}$, there two possibilities. If $\mathcal{P}$ has a single face, we define the terminators likewise: any two of the three children can be made terminators. If two terminators are chosen then $\mathcal{P}$ becomes *a terminated 2-path*. The terminators may share a vertex even if the 2-path in question has more than one face. Note that saying "$\mathcal{P}$ is a terminated 2-path with terminators that are single edges" is equivalent to saying that these edges are outer edges and $\mathcal{P}$ 2-connects them. Figure 6 illustates a terminated 2-path.

If we choose only one terminator then $\mathcal{P}$ is called *a semi-terminated 2-path*. In a semi-terminated 2-path the terminator is not considered a child of
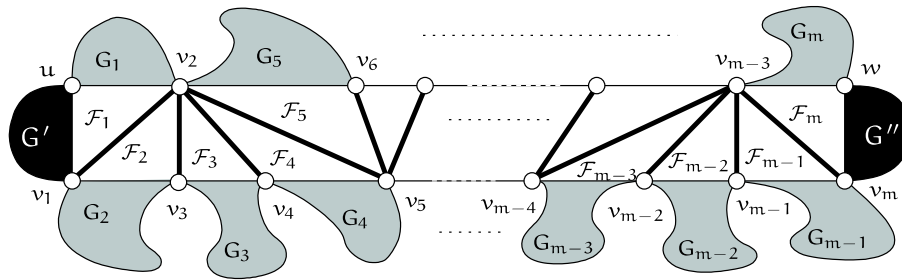
Fig. 6. The terminators of $\mathcal{P} = \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ are G' and G'' and they are drawn in black, while $\mathsf{G}_1, \mathsf{G}_2, \ldots, \mathsf{G}_m$ are the children of $\mathcal{P}$ and they are drawn in gray

the 2-path, analogously to the case of two terminators. A 2-path for which no terminators are chosen is *non-terminated*. It is obvious that a non-terminated 2-path has $m + 2$ children, a semi-terminated one has $m + 1$ children, and a terminated one has m children.

When we say "the children of a face $\mathcal{F}$", we mean the three children of the 2-path that consists of $\mathcal{F}$ only. The distinction between "the submops
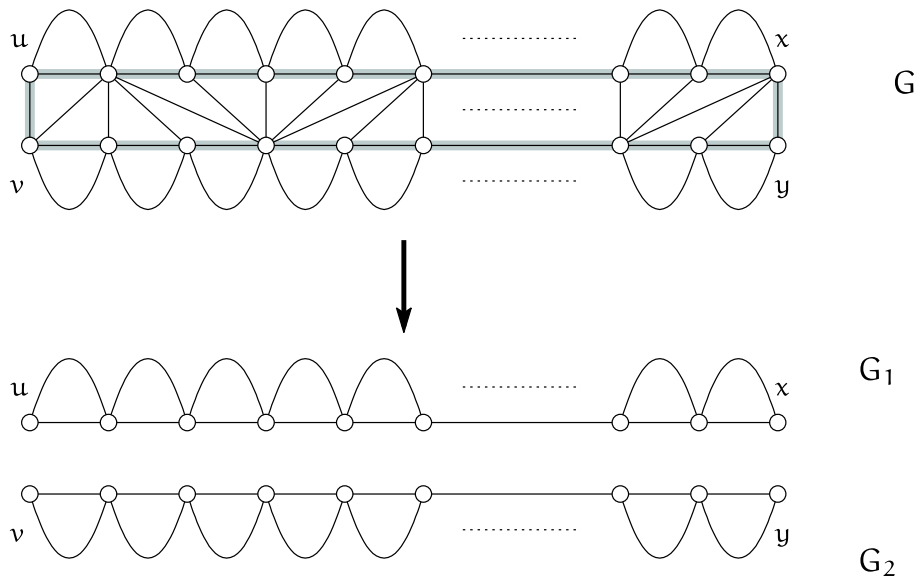

Fig. 7. $\mathcal{P}$ is the 2-path in G outlined by a thick gray line. Its severing is the removal of the outer edges (u, v) and (x, y) and all its inner edges. The severing leads to the connected components G_1 and G_2

begotten by a face" and "the children of a face" is that in the former case the three submops are not rooted, while in the second case they are rooted.

Let G be a mop, (u, v) and (x, y) be two outer edges that may or may not share an endpoint, and $\mathcal{P}$ be the 2-path that 2-connects (u, v) and (x, y). Remove (u, v), (x, y), and all the inner edges of $\mathcal{P}$ from G. As a result of this edge removal we get two connected components – call them G_1 and G_2. In general, none of them is a mop, being a graph with cut vertices. We say that G_1 *and* G_2 *are the components after severing* $\mathcal{P}$. We emphasise that the severing operation is applicable only to 2-paths that 2-connect two distinct outer edges.

2.7. Linear Layouts. *A linear layout of a graph* $G = (V, E)$, or simply *a layout*, is a bijective function $L : V \rightarrow \{1, 2, \ldots, |V|\}$. For any layout L and vertex u, $\pi_L(u)$ is defined to be $\pi_L(u) = \{v \in V \mid L(v) \leq L(u)$, and for some $w \in V$, $L(w) > L(u)$ and $(v, w) \in E\}$. *The vertex separation of* u *under* L *is* $|\pi_L(u)|$. For any $v \in \pi_L(u)$, we say that v *contributes* to the separation of u. *The vertex separation of* G *under* L *is* $vs_L(G) = \max_{u \in V}\{|\pi_L(u)|\}$ and *the vertex separation of* G *is* $vs(G) = \min\{vs_L(G) \mid L$ is a linear layout of $G\}$. Any layout L such that $vs_L(G) = vs(G)$ is called *optimal*. If for some $u \in G$, $|\pi_L(u)| = vs_L(G)$, we call u *heavy under L*, or simply *heavy* in case it is clear which layout we mean. If p is a path in G, we say that p *contributes to the separation of* u *under* L if at least one vertex from p is in $\pi_L(u)$.

For any $u, v \in G$ such that $L(u) < L(v)$, we say that under L, u is *left of* v, and v is *right of* u. Assume that $adj(u) \neq \emptyset$. We define left(u) to be the leftmost vertex among $adj(u)$ and $\overline{right}(u)$ to be the rightmost vertex among $adj(u)$, under L. The vertex *right*(u) is defined as follows: if u is left of $\overline{right}(u)$ then right$(u) = \overline{right}(u)$, else right$(u) = u$.

Throughout this paper we think of linear layouts as of lists of vertices, rather than as mappings of vertices on integers. If $G_1 = (V_1, E_1)$ is a subgraph of $G = (V, E)$, then *the sublayout of* G_1 *under* L is the list of the vertices from V_1, ordered in the same way as in L. For any $u \in V$, by "$L - u$" we denote the list L with u deleted from it and the other vertices left in the same relative order. For any $U \subset V$, by "$L - U$" we denote the list L with the vertices from U deleted and the other vertices left in the same relative order. If V is partitioned into V_1 and V_2, G_i is the subgraph induced by V_i, and L_i is a layout for G_i for $i = 1, 2$, we use the notation "$L = L_1, L_2$" to denote the layout L for G that is obtained by concatenation of the lists L_1 and L_2. The notation can be generalised for multiple layouts in the obvious way.

For any layout $L = u_1, u_2, \ldots, u_n$, *an interval in* L is a contiguous, possibly empty subsequence $u_i, u_{i+1}, \ldots, u_{i+j}$ of L.

Definition 5. *Let* $G = (V, E)$ *be a non-trivial mop. Any layout* L *for* G *is called* canonical *if* $\left(L^{-1}(1), L^{-1}(2)\right)$ *and* $\left(L^{-1}(|V| - 1), L^{-1}(|V|)\right)$ *are outer edges* G.

2.8. Stretchability, appendability, and affixability of layouts.

Definition 6 ([11], Definition 3)**.** *Suppose that* $G = (V, E)$ *is a graph and* U *and* W *are not necessarily disjoint subsets of* V, *each of them being possibly empty. Say,* $U = \{u_1, u_2, \ldots, u_n\}$ *and* $W = \{w_1, w_2, \ldots, w_m\}$. *Suppose that* L *is a layout of* G *and the leftmost and the rightmost vertices of* L *are* a *and* z, *respectively. Suppose that* $\mathcal{I}_{i,left}$ *is the possibly empty interval in* L *from and including* a *to and excluding* u_i, *for* $1 \leq i \leq n$. *Suppose that* $\mathcal{I}_{j,right}$ *is the possibly empty interval in* L *from and including* $right(w_j)$ *to and including* z, *for* $1 \leq j \leq m$. *Suppose that* $\mathcal{J}_{i,left}$ *is the possibly empty interval in* L *from and including* a *to and excluding* w_i, *for* $1 \leq i \leq m$. *Suppose that* $\mathcal{J}_{j,right}$ *is the possibly empty interval in* L *from and including* $right(u_j)$ *to and including* z, *for* $1 \leq j \leq n$. *We say that* L *is* k-stretchable with respect to U and W *if at least one of the following holds:*

- *The separation of any vertex in* L *is at most* k *minus the number of intervals from* $\mathcal{I}_{1,left}, \mathcal{I}_{2,left}, \ldots, \mathcal{I}_{n,left}, \mathcal{I}_{1,right}, \mathcal{I}_{2,right}, \ldots, \mathcal{I}_{m,right}$ *that it is in.*

- *The separation of any vertex in* L *is at most* k *minus the number of intervals from* $\mathcal{J}_{1,left}, \mathcal{J}_{2,left}, \ldots, \mathcal{J}_{m,left}, \mathcal{J}_{1,right}, \mathcal{J}_{2,right}, \ldots, \mathcal{J}_{n,right}$ *that it is in.*

In the former case, we say that U *is associated with the left direction and* W, *with the right one. In the latter case, we say the opposite.*

Stretchability with respect to two vertex sets is a generalisation of both stretchability with respect to two vertices (see Definition 5 on page 19 in [10]) and extensibility with respect to a single vertex (see Definition 2.2 on page 4 in [6]). Theorem 1 proves that stretchability with respect to two vertex sets is, to put it informally, left-right symmetrical. That allows us to talk about stretchability with respect to U and W without associating each vertex set with a certain direction.

If the graph G has a layout that is stretchable in a certain way, we say that G is stretchable in that way, too.

The importance of stretchability of graphs is based on the following observation, which follows easily from Definition 6.

Observation 1. *Under the naming convention of Definition 6, if we add* n *vertices not from* G *to the left side of* L *and connect each of them to a distinct*

vertex from U, *and we add one or more other vertices not from* G *to the right side of* L *and connect each vertex from* W *to one or more of them and we call the thus obtained layout* L^*, *then* L *is* k-*stretchable with respect to* U *and* W *so that* U *is associated with the left direction and* W, *with the right one, if and only if* L^* *has separation at most* k.

In the context of mops we give names to two particular types of stretchability.

Definition 7 (appendability and affixability)**.** *Suppose that* $G = (V, E)$ *is a mop,* $U, W \subset V$, *and* L *is a layout for* G *such that* L *is* k-*stretchable with respect to* U *and* W. *If* $U = \{u, v\}$, $W = \emptyset$, *and* (u, v) *is an outer edge of* G *we say that* L *is* k-*appendable with respect to* u *and* v. *If* $U = \{u\}$, $W = \{v\}$, *and* (u, v) *is an outer edge of* G *then we say that* L *is* k-*affixable with respect to* u *and* v. *If* G *has a layout that is appendable or affixable in a certain way, we say that* G *is appendable or affixable in that way, too.*

In case that G is a rooted mop and the root edge is (x, y), when we talk about appendability or affixability without specifying with respect to which vertices, we invariably mean with respect to x and y.

Lemma 2. *Suppose that* $G = (V, E)$ *is a connected graph. Suppose that* W *and* Z *are two not necessarily disjoint subsets of* V *and* L *is a layout for* G *that is* k-*stretchable with respect to* W *and* Z *so that* W *is associated with the left direction and* Z, *with the right one. If we modify* L *by placing the vertices from* W *to the leftmost* $|W|$ *positions while keeping the remaining vertices in the same order, the obtained layout is still* k-*stretchable with respect to* W *and* Z.

P r o o f. This is just a reformulation of [11, Lemma 1 on pp. 5]. □

Observation 2. *Suppose that* G *is a rooted mop,* (u, v) *is an outer egde in* G, L *is a layout for* G, *and* $k \geq 2$.

1. *If* L *is* k-*appendable with respect to* u *and* v *so that* u *and* v *are associated with the left direction, by Lemma 2 we can assume without loss of generality that* $L(u) = 1$ *and* $L(v) = 2$ *or the opposite. Then Definition 7 is equivalent to saying any vertex in* L *has separation* $\leq k$.

2. *If* L *is* k-*appendable with respect to* u *and* v *so that* u *and* v *are associated with the right direction, Definition 7 is equivalent to the following. Suppose* x *is the leftmost among* $right(u)$ *and* $right(v)$ *and* y *is the rightmost one,*

   (a) *If* $x \neq y$ *then the Definition 7 is equivalent to saying that the vertices left of, but not including,* x *have separation at most* k, *the vertices*

between x and y, including x and excluding y, have separation at most $k-1$, and the vertices right of and including y have separation at most $k-2$.

    (b) If $x = y$ then Definition 7 is equivalent to saying that the vertices left of, but not including, x have separation at most k, and the vertices right of and including x have separation at most $k-2$.

3. If L is k-affixable with respect to u and v, by Lemma 2 we can assume without loss of generality that $L(u) = 1$ and then Definition 7 is equivalent to saying that the vertices left of $right(v)$ have separation at most k, and the vertices right of and including $right(v)$ have separation at most $k-1$.

part 1: appendability, left direction

part 2: appendability, right direction

part 3: affixability, u in the left direction
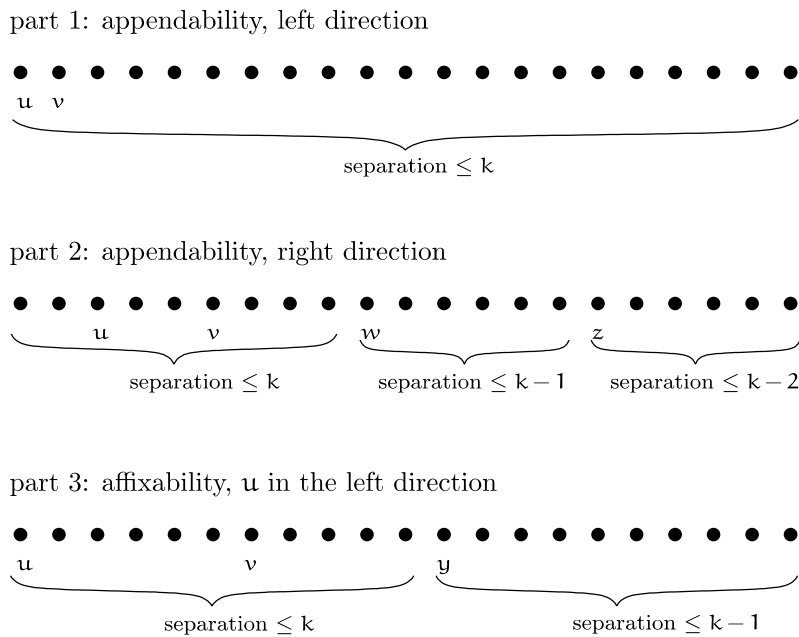
Fig. 8. Observation 2 on appendability and affixability. In part 2 w is the leftmost, and z is the rightmost, of $right(u)$ and $right(v)$. In part 3, $y = right(v)$

These observations follow trivially from Definition 6 and Definition 7. Figure 8 illustrates Observation 2. We do not illustrate Case 2b since Case 2a is more general.

Observation 3. *Under the assumptions of Observation 2, if* G *is* k-*affixable with respect to* u *and* v, *then* G *is* $(k+1)$-*appendable with respect to* u *and* v.

To see that the Observation 3 is true, look at Figure 8, part 3, and imagine v is moved to the second leftmost position without changing the relative positions of the other vertices. Clearly, that can increase the separations of the vertices that were left of v before the move, and right of v after the move, by at most one. So, the modified layout starts with u and v and has separation at most $k+1$. Apply Observation 2, Case 1, and the desired conclusion follows. □

Theorem 1 (Theorem 1 in [11], p. 6). *Suppose that* $G = (V, E)$ *is a connected graph. Suppose that* W *and* Z *are two not necessarily disjoint subsets of* V. *Then* G *has a layout that is* k-*stretchable with respect to* W *and* Z *where* W *is associated with the left direction and* Z, *with the right one, if and only if* G *has a layout that is* k-*stretchable with respect to* W *and* Z *where* W *is associated with the right direction and* Z, *with the left one.*

3. Constructions.

Construction 1. *Let* G *be a mop and* $e = (u, v)$ *be any inner edge in* G. *Let* G_1 *and* G_2 *be the two submops after the splitting of* e. *Let* L_1 *and* L_2 *be* k-*appendable with respect to* u *and* v *layouts for* G_1 *and* G_2, *respectively. The output is a layout* L *for* G, *such that* $vs_L(G) \leq k$.

By Theorem 1 we may assume without loss of generality that in L_1, the vertex set $\{u, v\}$ is associated with the right direction and in L_2, with the left one. By Lemma 2 we may assume without loss of generality that $L_2(u) = 1$ and $L_2(v) = 2$. Let $\overline{L_2} = L_2 - \{u, v\}$. We output $L = L_1, \overline{L_2}$. Next we show that $vs_L(G) \leq k$.

For any vertex $x \in \overline{L_2}$, $|\pi_{\overline{L_2}}(x)| \leq k$. That follows immediately from Observation 2, part 1. Now consider any vertex $x \in L_1$. The interval L_1 of L is partitioned into three subintervals (one of which can be empty) as suggested Observation 2, part 2 (see Figure 8, part 2). If x is in the leftmost of them its separation is $\leq k$ before $\overline{L_2}$ is added and does not increase after $\overline{L_2}$ is added. If x is in the middle interval then its separation is $\leq k - 1$ before $\overline{L_2}$ is added and increases by one after $\overline{L_2}$ is added since one of u, v that did not contribute to its separation before, contributes to it now. If x is in the rightmost of them its separation is $\leq k - 2$ before $\overline{L_2}$ is added and increases by precisely two after $\overline{L_2}$ is added since both u and v did not contribute to it before and contribute to it now. □

Construction 2. *Let* G *be a rooted mop with root edge* (u, w) *and root face* $\mathcal{F} = (u, v, w)$. *Let the child of* $\mathcal{F}$ *rooted at* (u, v) *be* G_1 *and the one rooted at*
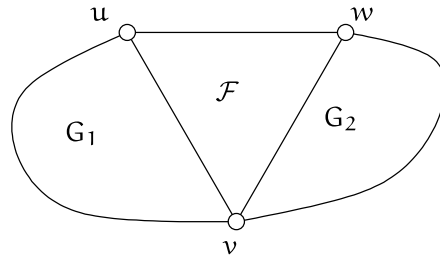
Fig. 9. The rooted mop from Construction 2

(v, w) *be* G_2. *Let* L_1 *be a* $(k-1)$-*affixable layout for* G_1 *and* L_2 *be a* k-*appendable layout for* G_2. *The output is a* k-*appendable layout for* G.

Figure 3 illustrates G. By Theorem 1 we may assume without loss of generality that in L_1, u is associated with the left direction and v, with the right one, and that in L_2, $\{v, w\}$ is associated with the left direction. By Lemma 2 we may assume that $L_1(u) = 1$ and that $L_2(v) = 1$ and $L_2(w) = 2$. Let $\overline{L_1} = L_1 - u$ and $\overline{L_2} = L_2 - \{v, w\}$. We output $L = u, w, \overline{L_1}, \overline{L_2}$. Next we show that L is k-appendable with respect to u and w so that $\{u, w\}$ is associated with the left direction. Consider any vertex $x \in \overline{L_1}$. By Observation 2, part 3, if x is between u and right(v) its separation under $\overline{L_1}$ is $\leq k - 1$, so under L it is $\leq k$ since $\pi_L(x) \setminus \pi_{\overline{L_1}}(x) = \{w\}$. If x is right of, or coincides with, right(v), its separation under $\overline{L_1}$ is $\leq k - 2$, so under L it is $\leq k$ because $\pi_L(x) \setminus \pi_{\overline{L_1}}(x) = \{v, w\}$. Now consider any vertex x in $\overline{L_2}$. Clearly, $\pi_L(x) = \pi_{\overline{L_2}}(x)$, therefore $\left| \pi_{\overline{L}}(x) \right| \leq k$.

Having proven that all vertices in $\overline{L_1}$ and $\overline{L_2}$ have separation $\leq k$, apply Observation 2, part 1, to conclude L is k-appendable with respect to u and w so that u and w are associated with the left direction. $\square$

Construction 3. *Let* G *be a rooted mop with root edge* (u, w) *and root face* $\mathcal{F} = (u, v, w)$. *Let the child of* $\mathcal{F}$ *rooted at* (u, v) *be* G_1 *and the one rooted at* (v, w) *be* G_2. *Let* L_1 *be a* $(k-1)$-*affixable layout for* G_1 *and* L_2 *be a* $(k-1)$-*affixable layout for* G_2. *The output is a* k-*appendable layout for* G.

G looks exactly as the mop from Construction 2. By Observation 3, the layout $\widetilde{L_2}$ obtained from L_2 by moving w to the second position without changing the relative positions of the other vertices is k-appendable with respect to v and w. Apply Construction 2 on L_1 and $\widetilde{L_2}$. $\square$

Construction 4. *Let* $G = (V, E)$ *be a mop with at least two faces. The input is a layout* L *for* G *of separation* $\leq k$. *The output is a layout for* G *of separation* $\leq k$ *such that the two leftmost vertices are adjacent.*

Summary of the construction: if the two leftmost vertices are not adjacent, move $L^{-1}(1)$ immediately right of $\text{left}(L^{-1}(1))$ until it is the case that the two leftmost vertices are adjacent.

Let $L^{-1}(1) = a$ and $L^{-1}(2) = b$. If $(a, b) \in E$ we are done so assume $(a, b) \notin E$. Let $w_a = \text{left}(a)$. But a is a mop vertex, so $|\text{adj}(a)| \geq 2$, therefore $\exists z_a \in \text{adj}(a)$ such that $L(z_a) > L(w_a)$:

$$L = a, \underbrace{b, \dots\dots\dots\dots\dots}_{\text{interval } \mathcal{I}_1}, w_a, \underbrace{\dots\dots\dots, z_a, \dots\dots\dots}_{\text{interval } \mathcal{I}_2}$$

We modify L by placing a immediately right of w_a without changing anything else. Call that layout L_1.:

$$L_1 = \underbrace{b, \dots\dots\dots\dots\dots}_{\text{interval } \mathcal{I}_1}, w_a, a, \underbrace{\dots\dots\dots, z_a, \dots\dots\dots}_{\text{interval } \mathcal{I}_2}$$

Consider how this change affects the separation of the layout.

- For any vertex $x \in \mathcal{I}_1$, $|\pi_{L_1}(x)| = |\pi_L(x)| - 1$ because $a \in \pi_L(x)$ and $a \notin \pi_{L_1}(x)$.

- Consisder w_a. Note that $a \in \pi_L(w_a)$ because $(a, z_a) \in E$. Obviously, $a \notin \pi_{L_1}(w_a)$ and $w_a \in \pi_{L_1}(w_a)$. If $w_a \in \pi_L(w_a)$ then $\pi_{L_1}(w_a) = \pi_L(w_a) \setminus \{a\}$, therefore $|\pi_{L_1}(w_a)| = |\pi_L(w_a)| - 1$. Otherwise $\pi_{L_1}(w_a) = (\pi_L(w_a) \setminus \{a\}) \cup \{w_a\}$, so $|\pi_L(w_a)| = |\pi_{L_1}(w_a)|$.

- Consider a. Clearly, $\pi_{L_1}(a) = (\pi_L(w_a) \setminus a) \cup \{a\}$, *i.e.* $|\pi_{L_1}(a)| = |\pi_L(w_a)|$.

- For any vertex $x \in \mathcal{I}_2$, $|\pi_{L_1}(x)| = |\pi_L(x)|$.

We conclude that the moving of a does not increase the separation of any vertex in the layout. However, in L_1 there is at least one pair of consecutive adjacent vertices, namely w_a, a. Let $\delta_1 = L_1(w_a)$. Let c be the second leftmost vertex in L_1:

$$L_1 = b, c, \dots\dots\dots\dots\dots, w_a, a, \dots\dots\dots$$

If $(b, c) \in E$, we are done. Else let $w_b = \text{left}(b)$ with respect to L_1. It is possible to be the case that $L_1(w_b) < L_1(w_a)$ or $L_1(w_a) < L_1(w_b)$ or $L_1(w_b) = L_1(w_a)$, if $w_a = w_b$. Move b immediately right of w_b, obtaining some layout L_2. For the same reasons as above this modification does not increase the separation, even if $w_a = w_b$. Let $\delta_2 := \min(L_2(w_a), L_2(w_b))$. That is how L_2 looks like:

$$L_2 = c, \dots\dots\dots, w_b, b, \dots\dots\dots$$

Note that $\delta_2 < \delta_1$. Therefore, in L_2 it is guaranteed there is a pair of adjacent vertices that is closer to the left side than the pair of adjacent vertices guaranteed to exist in L_1. That pair in L_2 can be w_a, a or w_b, b, that does not matter.

Now it is easy to see an iterative procedure that operates on our layout, at each successive step moving the leftmost vertex immediately right of the leftmost vertex it is adjacent to, without increasing the separation of any vertex, until the leftmost and the second leftmost vertex are adjacent. The procedure will certainly terminate because:

- after each step we create a pair of consecutive adjacent vertices in the layout,

- the leftmost such pair we created is closer to the left end of the layout with each step, and

- the layout is finite.   □

Lemma 3. *Under the assumptions of Construction 4, let u and v be any two vertices in G. Assume L is k-appendable with respect to u and v so that they are associated with the right direction. The layout L^* output by Construction 4 is k-appendable in the same way.*

P r o o f. Let $\mathcal{U}$ and $\mathcal{V}$ denote, with respect to the current layout we consider, the interval consisting of the vertices right of, or coinciding with, $\mathrm{right}(u)$, and the interval consisting of the vertices right of, or coinciding with, $\mathrm{right}(v)$, respectively. The layout being k-appendable so that u and v are associated with the right direction is equivalent to:

- the vertices outside $\mathcal{U}$ and $\mathcal{V}$ having separation at most k and either

  ○ the vertices in $\mathcal{U}$ having separation at most $k-1$ and those in $\mathcal{V}$ having separation at most $k-2$, or

  ○ the vertices in $\mathcal{U}$ having separation at most $k-2$ and those in $\mathcal{V}$ having separation at most $k-1$.

Without loss of generality, assume that in L the vertices from $\mathcal{U}$ have separation at most $k-1$ and those in $\mathcal{V}$ have separation at most $k-2$. In the justification of Construction 4 we proved that the separation of any vertex is $\leq k$ after any iteration, so it is enough to show that in L^* the vertices from $\mathcal{U}$ have separation at most $k-1$ and those in $\mathcal{V}$ have separation at most $k-2$. We do the proof for $\mathcal{U}$ only since the proof for $\mathcal{V}$ is completely analogous.

Let $d = \overline{\text{right}}(u)$ in L. First suppose $L(u) < L(d)$. Then $d = \text{right}(u)$ in L. Furhermore, $d = \text{right}(u)$ after any vertex move performed by Construction 4. To see why that is true, consider that $|\text{adj}(u)| \geq 2$ and let d' be the leftmost vertex of $\text{adj}(u) \setminus \{d\}$. Clearly, $L(d') < L(d)$.

- If $L(u) < L(d')$ then even if u is moved, it will be placed immediately right of d' and neither d' nor u will be moved by further iterations of Construction 4—if d' ever becomes the leftmost vertex of the current layout, the iterations stop—so d will always be right of u.

- If $L(d') < L(u)$ then u will never be moved by Construction 4 because u cannot become the leftmost vertex unless d' is moved; however, if d' is moved then it will be immediately right of a vertex that is either u or another vertex from $\text{adj}(d')$ and if ever that vertex, u or not, becomes the leftmost one, the iterations stop, therefore d will always be right of u.

So, under the assumption that $L(u) < L(d)$, it is invariantly true at any iteration of Construction 4 that $d = \text{right}(u)$. Therefore, at any iteration, $\mathcal{U}$ is the interval of the current layout consisting of the vertices right of and including d. Indeed, $\mathcal{U}$ can change because new vertices can be put in it. But in the justification of Construction 4 we proved that any vertex that has just been moved by the construction cannot have bigger separation than the vertex immediately left of it. It follows that the separation of the new vertices put in $\mathcal{U}$, if any, does not go beyond $k - 1$, and thus the separation throughout $\mathcal{U}$ when the construction finishes is bound by $k - 1$.

It remains to consider the possibility that $L(d) < L(u)$. So $u = \text{right}(u)$ in L. In this case u is never moved by the construction but it is possible $\text{right}(u)$ to become another vertex, only in case that one or more vertices from $\text{adj}(u)$ are moved right of u. If verices from $\text{adj}(u)$ are moved right of u, they will be moved (each in its turn) immediately right of u and so the rightmost of them, call it d'', will be $\text{right}(u)$ at the end of the iterations. But since $u = \text{right}(u)$ in L, it must be the case that the separation of u in L is at most $k - 1$. We conclude that any vertex from $\text{adj}(u)$ that is placed immediately right of u has new separation at most $k - 1$ and that remains so at any further iteration. Furthermore, any other vertex, *i.e.* a vertex not from $\text{adj}(u)$, that is moved to the right of u has separation after the moving at most that of the vertex immediately right of which it is placed and the latter vertex is from $\mathcal{U}$ in L. It follows that the separations of the vertices from $\mathcal{U}$ in L^* are at most $k - 1$. $\square$

Construction 5. *Let $G = (V, E)$ be a mop with at least two faces. The input is a layout L for G of separation $\leq k$. The output is a layout for G of separation $\leq k$ such that the two rightmost vertices are adjacent.*

Summary of the construction: if the two rightmost vertices are not adjacent, move $L^{-1}(|V|)$ immediately left of $\overline{\text{right}}(L^{-1}(|V|))$ until it is the case that the two rightmost vertices are adjacent.

Let the rightmost and second rightmost vertex in L be z and y, respectively. If $(y, z) \in E$ we are done so assume $(y, z) \notin E$. Let $a_z = \overline{\text{right}}(z)$:

$$L = \underbrace{\dots\dots\dots}_{\text{interval } \mathcal{J}_1}, a_z, \underbrace{\dots\dots\dots}_{\text{interval } \mathcal{J}_2}, y, z$$

We modify L by placing z immediately left of a_z without changing anything else. Call that layout L_1:

$$L_1 = \underbrace{\dots\dots\dots}_{\text{interval } \mathcal{J}_1}, z, a_z, \underbrace{\dots\dots\dots}_{\text{interval } \mathcal{J}_2}, y$$

Consider how this change affects the separation of the layout.

- For any vertex $x \in \mathcal{J}_2$, $|\pi_{L_1}(x)| \leq |\pi_L(x)|$.

- The separation of a_z either stays the same, in case there is a vertex in the interval $\mathcal{J}_2$ that a_z is connected to, or decreases by one, otherwise.

- Consider z. Clearly, $\pi_{L_1}(z) = (\pi_L(a_z) \setminus \{a_z\}) \cup \{z\}$. Note that $a_z \in \pi_L(a_z)$ since $(a_z, z) \in E$ by construction, so $|\pi_L(a_z) \setminus \{a_z\}| = |\pi_L(a_z)| - 1$. Therefore $|\pi_{L_1}(z)| = |\pi_L(a_z)|$.

- For any vertex $x \in \mathcal{J}_1$, $|\pi_{L_1}(x)| = |\pi_L(x)|$.

We conclude that the moving of z does not increase the separation of any vertex in the layout beyond k. However, in L_1 there is at least one pair of consecutive adjacent vertices, namely z, a_z. Let $\delta_1 = |V| - L_1(a_z)$. Let x be the second rightmost vertex in L_1:

$$L_1 = \dots\dots\dots, z, a_z, \dots\dots\dots, x, y.$$

If $(x, y) \in E$, we are done. Else let $a_y = \overline{\text{right}}(y)$ with respect to L_1. It is possible to be the case that $L_1(a_z) < L_1(a_y)$ or $L_1(a_y) < L_1(a_z)$ or $L_1(a_y) = L_1(a_z)$, if $a_y = a_z$. We move y immediately left of a_y, obtaining some layout L_2. For the

same reasons as above this modification does not increase the separation, even if $a_y = a_z$. Let $\delta_2 := \min\left(|V| - L_2(a_y), |V| - L_2(a_z)\right)$. That is how L_2 looks like:

$$L_2 = \ldots\ldots\ldots, y, a_y, \ldots\ldots\ldots, x.$$

Note that $\delta_2 < \delta_1$. Therefore, in L_2 it is guaranteed there is a pair of adjacent vertices that is closer to the right side than the pair of adjacent vertices guaranteed to exist in L_1. That pair in L_2 can be z, a_z or y, a_y, that does not matter.

Now it is easy to see an iterative procedure that operates on our layout, at each successive step moving the rightmost vertex immediately left the rightmost vertex it is adjacent to, without increasing the separation of any vertex, until the rightmost and the second rightmost vertex are adjacent. The procedure will certainly terminate because:

- after each step we create a pair of consecutive adjacent vertices in the layout,

- the rightmost such pair we created is closer to the right end of the layout with each step, and

- the layout is finite.   □

Lemma 4. *Under the assumptions of Construction 5, let* $L^{-1}(1) = u$, $L^{-1}(2) = v$, *and* $(u, v) \in E$ *be an outer edge. Then* u *and* v *are the leftmost and second leftmost vertex, respectively, of* L^*, *where* L^* *is the layout output by Construction 5.*

P r o o f. Obviously, Construction 5 cannot move any vertex left of u. We argue that Construction 5 does not move any vertex left of v either. Assume that some vertex d is placed immediately left of v by the construction, *i.e.* the layout becomes $u, d, v, \ldots$. Then v has to be $\overline{\text{right}}(d)$ in L. But we know that $|\text{adj}(d)| \geq 2$ and so at least one other vertex from $\text{adj}(d)$ must be left of v. That vertex can only be u. It follows that $\text{adj}(d) = \{u, v\}$. Recall that (u, v) is an outer edge so there can be only one vertex adjacent to both u and v. It follows that G is a 3-clique, *i.e.* G has a single face, with $V(G) = \{u, v, d\}$. But that contradicts the assumption that G has at least two faces. Since no vertex is moved immediately left of v, L^* starts with u and v too: $L^* = u, v, \ldots$ □

4. Lemmas and Theorems.

Lemma 5. *Let* $G = (V, E)$ *be a mop with at least two faces such that* $vs(G) \leq k$. *There exists a layout for* G *of separation* $\leq k$ *such that the two leftmost vertices are adjacent and the two rightmost vertices are adjacent.*

P r o o f. Starting with any optimal layout for G, apply Constructions 4 and 5. The two constructions are clearly independent in the sense that applying the second one does not cause violation of the desirable properties of the first one, namely separation $\leq k$ of the whole layout and the two vertices at one extreme being from the same edge. $\square$

Lemma 6. *Let* $G = (V, E)$ *be a mop with at least two faces such that* $vs(G) \leq k$. *There exists a canonical layout for* G *of separation* $\leq k$.

P r o o f. By Lemma 5 there exists a layout L for G such that $vs_L(G) \leq k$ and, if the two leftmost vertices in L are u and v and the two rightmost vertices are x and y, then $(u, v) \in E$ and $(x, y) \in E$:

$$L = u, v, \ldots\ldots\ldots\ldots\ldots, x, y.$$

However, Lemma 5 does not guarantee that (u, v) or (x, y) is an outer edge.

Assume G is as suggested by Figure 10: G' is the submop after splitting (u, v) that does not contain x and y, D' is the submop after splitting (x, y) that does not contain u and v, and $\mathcal{P}$ is the terminated 2-path with terminators G' and D' that 2-connects (u, v) with (x, y).
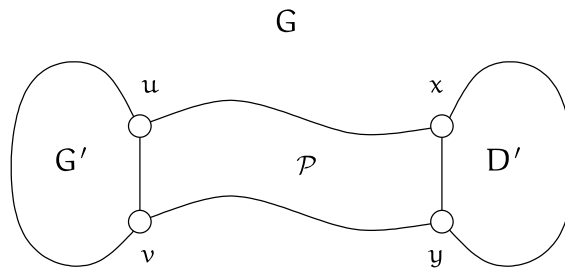

Fig. 10. The mop in the proof of Lemma 6. The mop is shown schematically as a terminated 2-path $\mathcal{P}$ with terminators G' and D' that 2-connects the inner edges (u, v) and (x, y). The children of $\mathcal{P}$ are not shown

Part I: the left side Define that *the hot edge* is the edge whose vertices are the two leftmost vertices of the layout for G we currently consider. So, (u, v) is the hot edge in L.

If G' is trivial, *i.e.* if the hot edge is an outer edge, we are done with the left side of L. So assume the hot edge is an inner edge. Consider G' and $G \ominus G'$. Let L' and $\widetilde{L}$ be the sublayouts of G' and $G \ominus G'$, respectively, under L. As both L' and $\widetilde{L}$ start with u and v and have separations $\leq k$, both must be

k-appendable with respect to u and v layouts, such that u and v are associated with the left direction. If G' is 3-clique, say $V(G') = \{u, v, d\}$, then obviously the layout $d, \widetilde{L}$ is a layout for G that starts with the vertices of an outer edge, namely (d, u), and has separation no more than k, so we are done with the left side. So assume that G' has at least two faces.

By Theorem 1 applied to G' and L', there exists a layout L^* for G' that is k-appendable with respect to u and v so that in it u and v are associated with the right direction. Apply Construction 4 to L^* to modify into another layout of separation at most k, call it L^{**}, that starts with some adjacent vertices u_1 and v_1. By Lemma 3, L^{**} is k-appendable with respect to u and v so that in it u and v are associated with the right direction.

Both L^{**} and $\widetilde{L}$ are k-appendable with respect to u and v layouts for the submops of G, obtained after splitting (u, v). Using L^{**} and $\widetilde{L}$, Construction 1 builds a layout L_1 for G of separation $\leq k$. L_1 has the additional property of having u_1 and v_1 as the two leftmost vertices. If (u_1, v_1) is an outer edge, we are done with the left side.
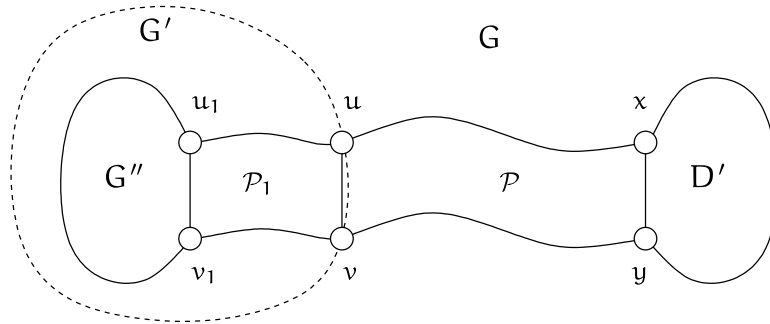

Fig. 11. The essence of Part I of the proof of Lemma 6. Since G'' is a proper submop of G', we effectively augment $\mathcal{P}$ with $\mathcal{P}_1$ when going from L to L_1 as the layout for G under consideration

Otherwise, let G'' be the submop of G after splitting (u_1, v_1) that does not contain x, y, u, and v (see Figure 11. Note that G'' is a proper submop of G'. Substitute L with L_1 as the layout under consideration for G and augment $\mathcal{P}$ with the 2-path $\mathcal{P}_1$, which 2-connects (u_1, v_1) with (u, v) (Figure 11). With respect to L_1, (u_1, v_1) is the hot edge. Proceed "inside" G'' exactly in the same manner.

We can iterate like that only a finite number of times before we get a hot edge that is an outer edge since G is finite and we "never go back", *i.e* the augmented 2-path that connects the hot edge with (x, y) gets longer and longer.

When we get a hot edge, say (a, b), as an outer edge, we are done with the left side of the layout of G. Call $\overline{L}$ the obtained layout, *i.e.* the layout for G of separation $\leq k$ such that $\overline{L}(1) = a$, $\overline{L}(2) = b$, and (a, b) is an outer edge.

Part II: the right side Let the hot edge be the edge whose vertices are the two rightmost vertices of the layout for G we currently consider. So, (x, y) is the hot edge in $\overline{L}$. Call $\overline{\mathcal{P}}$ the 2-path that 2-connects (a, b) with (x, y). Note that (a, b) and D' are the terminators of $\overline{\mathcal{P}}$.

If D' is trivial, *i.e.* if the hot edge is an outer edge, we are done with the right side of $\overline{L}$. So assume the hot edge is an inner edge. Consider D' and $G \ominus D'$. Let M' and $\widetilde{M}$ be the sublayouts of D' and $G \ominus D'$, respectively, under $\overline{L}$. Note that both M' and $\widetilde{M}$ are k-appendable with respect to x and y layouts so that x and y are associated with the right direction, because y is the rightmost vertex in both of them and so its separation is 0; furthermore, in both of them $y = \text{right}(y)$ and, since $(x, y) \in E$, $y = \text{right}(x)$. If D' is a 3-clique, say $V(D') = \{x, y, d\}$, then obviously the layout $\widetilde{M}, d$ is a layout for G of separation no more than k whose two rightmost vertices, namety y and d, are the endpoints of an outer edge, so we are done with the right side. So assume that D' has at least two faces.

By Theorem 1 applied to D' and M', there exists a layout M^* for D' that is k-appendable with respect to x and y so that in it x and y are associated with the left direction. Apply Construction 5 to M^* to modify into another layout of separation at most k, call it M^{**} whose two rightmost vertices are some adjacent vertices, say x_1 and y_1. By Lemma 4, M^{**} starts with x and y just like M^*, therefore M^{**} is k-appendable with respect to x and y so that they are associated with the left direction.

Both M^{**} and $\widetilde{M}$ are k-appendable with respect to x and y layouts for the submops of G, obtained after splitting (x, y). Using M^{**} and $\widetilde{M}$, Construction 1 builds a layout $\overline{L_1}$ for G of separation $\leq k$. $\overline{L_1}$ has the additional property of having x_1 and y_1 as the two rightmost vertices. If (x_1, y_1) is an outer edge, we are done with the right side.

Otherwise, let D'' be the submop of G after splitting (x_1, y_1) that does not contain u, v, x, and y. Note that D'' is a proper submop of D'. Substitute $\overline{L}$ with $\overline{L_1}$ as the layout under consideration for G. With respect to $\overline{L_1}$, (x_1, y_1) is the hot edge. Proceed "inside" D'' exactly in the same manner.

We can iterate like that only a finite number of times before we get a hot edge that is an outer edge since G is finite and we "never go back". When we get a hot edge, say (z_1, z_2), as an outer edge, we are done with the right side of the layout of G too. We built a layout for G of separation $\leq k$ that starts with a and b and finishes with z_1 and z_2, where (a, b) and (z_1, z_2) are outer edges. $\square$

Lemma 7. *Suppose* G *is a rooted mop with root edge* (u, v). *Suppose that* G *is* k-*affixable. Then for each inner edge* (x, y) *in* G, $G[x, y]$ *is* k-*affixable.*

P r o o f. Let L be a k-affixable layout for G. Without loss of generality assume that in L, u is associated with the left direction and v, with the right one. Without loss of generality, assume that x is peripherally closer to u (and so y is peripherally closer to v). Let α and ω be two vertices not from G. Let $G^* = (V \cup \{\alpha, \omega\}, E \cup \{(u, \alpha), (v, \omega)\})$ and $L^* = \alpha, L, \omega$. By Observation 1, L is k-stretchable with respect to u and v so that u is associated with the left direction and v, with the right one, if and only if $vs_{L^*}(G^*) \leq k$.

Let L_1 be the sublayout of $G[x, y]$ under L (and thus under L^*, too). Let x' be the vertex in G^*, such that:

- if $x \neq u$ then x' is the vertex from the peripheral path between u and x that is adjacent to x,

- else $x' = \alpha$.

Let y' be the vertex in G^*, such that:

- if $y \neq v$ then y' is the vertex from the peripheral path between y and v that is adjacent to y,

- else $y' = \omega$.

Let $G' = (V(G[x, y]) \cup \{x'y'\}, E(G[x, y]) \cup \{(x, x'), (y, y')\})$. Note that G' is a subgraph of G^*, though not necessarily a subgraph of G. Let the sublayout of G' under L^* be L'. Since G' is a subgraph of G^*, $vs_{L'}(G') \leq k$. But Observation 1 applied to $G[x, y]$ and G' says that $G[x, y]$ is k-stretchable with respect to x and y so that x is associated with the left direction and y, with the right one, if and only if L' has separation at most k. $\square$

Lemma 8. *Let* G *be rooted mop with root face* $\mathcal{F} = (u, v, w)$ *and root edge* (u, v) *and for each face in* G, *at least one child is* $(k-1)$-*affixable. Then* G *is* k-*appendable.*

P r o o f. First we build a semi-terminated 2-path with terminator (u, v) such that all its children are $(k-1)$-affixable. Let $\mathcal{F}^*$ be a variable of type face and $\mathcal{P}$ be a variable of type 2-path.

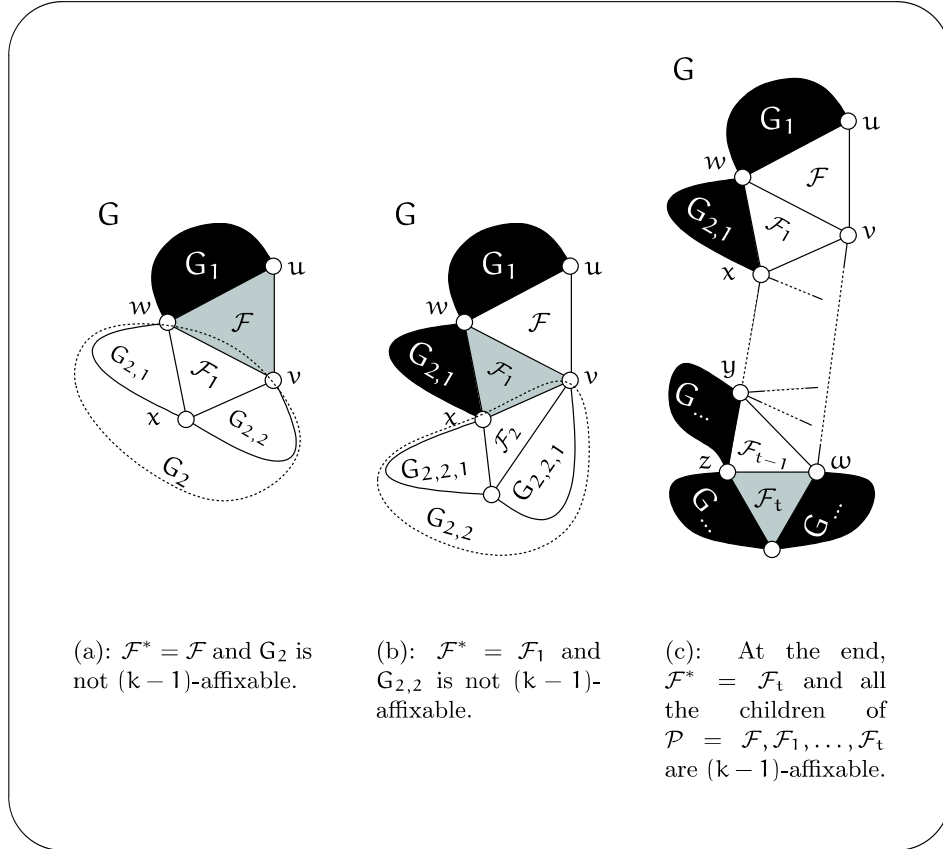**Initialise** Set $\mathcal{P} := \mathcal{F}$ and $\mathcal{F}^* := \mathcal{F}$.

Fig. 12. Building the 2-path whose children are $(k-1)$-affixable in the proof of Lemma 8. The children of $\mathcal{F}$ are G_1 and G_2, the children of $\mathcal{F}_1$ are $G_{2,1}$ and $G_{2,2}$, the children of $\mathcal{F}_2$ are $G_{2,2,1}$ and $G_{2,2,2}$, and so on. Initialising $\mathcal{F}^*$ to $\mathcal{F}$—see (a)—we set $\mathcal{F}^*$ to the root of the child that is not $(k-1)$-stretchable—see (b)—until we reach a face whose both children are $(k-1)$-stretchable (face $\mathcal{F}_t$ in (c)). The $(k-1)$-stretchable children are drawn in black and the current $\mathcal{F}^*$ is grayed. G_2 is the child of $\mathcal{F}$ that is not $(k-1)$-affixable, $G_{2,2}$ is the child of $\mathcal{F}_1$ that is not $(k-1)$-affixable, ..., $G[z, \omega]$ is the child of $\mathcal{F}_{t-1}$ that is not $(k-1)$-affixable, and both children of $\mathcal{F}_t$ are $(k-1)$-affixable

Iterate   By the statement of the lemma, at least one child of $\mathcal{F}^*$ is $(k-1)$-affixable. If both children of $\mathcal{F}^*$ are $(k-1)$-affixable, stop iterating. Otherwise, let G^* be the child that is not $(k-1)$-affixable. Set $\mathcal{F}^*$ to the root face of G^*, set $\mathcal{P} := \mathcal{P}, \mathcal{F}^*$ and continue iterating.

Finish   Return $\mathcal{P}$.

Figure 12 illustrates the work of the iterative procedure. The procedure is guaranteed to finish because:

- G is finite and, as we know from Lemma 7, the affixability of a child of a mop cannot exceed the affixability of the whole mop;

- the iterative procedure "never goes back", *i.e.* it goes only deeper and deeper "into" the rooted mop.

- the trivial mop is 1-affixable and if the iterations continue long enough, we will get to a face whose both children are edges, *i.e.* trivial mops.

Having built the said $\mathcal{P}$, we build a k-appendable with respect to u and v layout L for G. Assume the naming convention suggested by Figure 12. face—it has to be $\mathcal{F}$—use Construction 3 to build a k-appendable with respect to u and v layout for G. Otherwise, let $\mathcal{F}_t$ be the endface of $\mathcal{P}$ that is not $\mathcal{F}$ (see Figure 12(c)). Use Construction 3 to build a k-appendable with respect to z and w layout L_t for $G[\mathcal{F}_t]$, and then use t times Construction 2 as follows:

- using L_t and a $(k-1)$-affixable layout with respect to y and z layout for $G[y, z]$, build a k-apendable with respect to y and w layout L_{t-1} for $G[\mathcal{F}_{t-1}]$

- . . .

- using L_2—a k-appendable with respect to x and v layout for $G_{2,2}$—and a $(k-1)$-affixable with respect to x and w layout for $G_{2,1}$, build a k-apendable with respect to v and w layout L_1 for $G[\mathcal{F}_1]$,

- using L_1 and a $(k-1)$-affixable layout with respect to w and u layout for G_1, build a k-apendable with respect to u and v layout for $G[\mathcal{F}]$, *i.e.* for G. $\square$

Theorem 2. *For any mop* G, $vs(G) \leq k$ *if and only if for any face* $\mathcal{F}$ *in* G, *at least one child of* $\mathcal{F}$ *is* $(k-1)$-*affixable.*

Proof of sufficiency.   Assume that any face in G has a $(k-1)$-affixable child. We show how to build a layout L for G of separation at most k.

Let $\mathcal{F} = (u, v, w)$. Let $G[v, w]$ be a $(k-1)$-affixable child of $\mathcal{F}$ and let L_1 be a $(k-1)$-affixable layout for it. For any of the rooted mops $G[u, v]$ and $G[u, w]$, either it is $(k-1)$-affixable in which case by Observation 3 it is k-appendable, or it is not $(k-1)$-affixable but then it is k-appendable by Lemma 8. So, there exists a k-appendable layout L_2 for $G[u, v]$ and a k-appendable layout L_3 for $G[u, w]$. Using L_1, L_2, and Construction 2, build a k-appendable with respect to u and w layout L' for $G \ominus G[u, w]$. Then using L_3, L', and Construction 1, build a layout L for G such that $vs_L(G) \le k$.

P r o o f o f n e c e s s i t y. Assume that $vs(G) \le k$. By Lemma 6, there is canonical layout L for G of separation $\le k$. Let a and b be the two leftmost vertices, and y and z be the two rightmost vertices, of L. Let $\mathcal{P}$ be the 2-path that 2-connects (a, b) with (y, z).

Part I. Assume that $\mathcal{F}$ is a face in $\mathcal{P}$. Let G_1 be the child of $\mathcal{P}$ rooted at an edge of $\mathcal{F}$. We prove that G_1 is $(k-1)$-affixable. Let the root edge of G_1 be (u, v). Let G' and G'' be the components after severing $\mathcal{P}$. Without loss of generality assume that a, y, and G_1 are in G', and that in the path in G' connecting a and y, u is between a and v (thus v is between u and y). See Figure 13 for illustration of G. Let p'_u and p'_v be the paths in G' running between a and u, and between v and y, respectively. Certainly, a may coincide with u and y may coincide with v, so any of p'_u and p'_v may have length one. Let p'' be the path running between
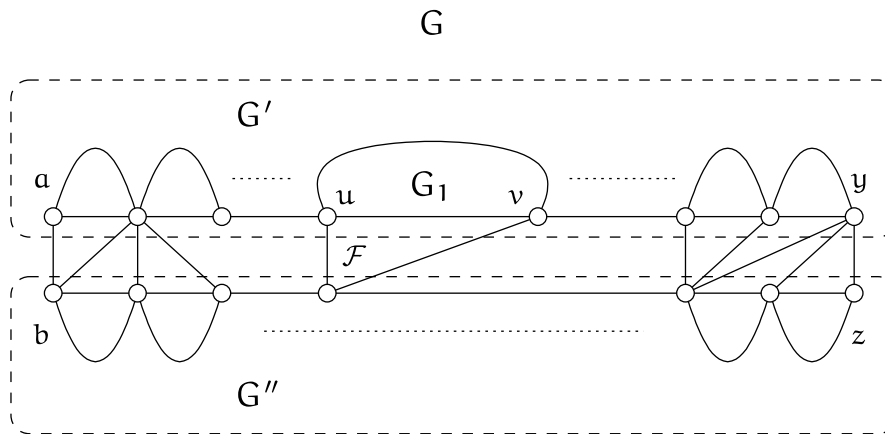

Fig. 13. The mop G from the proof of sufficiency, part I, of Theorem 2. $\mathcal{P}$ is the 2-path that 2-connects (a, b) and (y, z). G' and G'', the components after severing $\mathcal{P}$, are outlined with dashed lines. The face $\mathcal{F}$ is in $\mathcal{P}$. We prove that G_1, the child of $\mathcal{P}$ rooted at $\mathcal{F}$, is $(k-1)$-affixable

b and z in G''. Note that p'_u, p'_v, and p'' are pairwise vertex disjoint. Let L_1 be the sublayout of G_1 under L. We argue that L_1 is $(k-1)$-affixable with respect to u and v.

Part I.1.   Assume that $L(u) < L(v)$ (thus $L_1(u) < L_1(v)$). Let $\mathcal{I}_1$ be the interval of L_1 left of u but excluding u, and $\mathcal{I}_2$ be the interval of L_1 right of and including $\mathrm{right}(v)$. So L_1 looks like:

$$L_1 = \underbrace{\dots\dots\dots\dots}_{\text{interval } \mathcal{I}_1}, u, \dots\dots\dots, v, \dots\dots\dots, \underbrace{\mathrm{right}(v), \dots\dots\dots\dots}_{\text{interval } \mathcal{I}_2}$$

Certainly v may coincide with $\mathrm{right}(v)$ but that does not alter the validity of our proof. If $v = \mathrm{right}(v)$ we have:

$$L_1 = \underbrace{\dots\dots\dots\dots}_{\text{interval } \mathcal{I}_1}, u, \dots\dots\dots\dots, \underbrace{v, \dots\dots\dots\dots}_{\text{interval } \mathcal{I}_2}$$

If $\mathcal{I}_1$ is not empty consider any vertex w in it. If $\mathcal{I}_1$ is not empty, $a \neq u$ and so $L(a) < L(w)$ and $L(b) < L(w)$. Therefore, there is a vertex from p'_u and another vertex from p'', such that both those vertices are in $\pi_L(w)$ and not in $\pi_{L_1}(w)$. Since $|\pi_L(w)| \leq k$, it follows that $|\pi_{L_1}(w)| \leq k-2$.

Consider any vertex w from L_1, such that $w \notin \mathcal{I}_1$ and $w \notin \mathcal{I}_2$. If $w \neq u$, or if $w = u$ and $\mathcal{I}_1$ is not empty, then it must be the case that a vertex from p'' is in $\pi_L(w)$ and not in $\pi_{L_1}(w)$, so $|\pi_{L_1}(w)| \leq k-1$. If $w = u$ and $\mathcal{I}_1$ is empty it can be the case that no vertex from p'' contributes to the separation of w in L. But that can be true only if $u = a$ and $L(a) = 1$, *i.e.* in case that w is the leftmost vertex of both L and L_1. But then $|\pi_{L_1}(w)| = 1$ and since $k \geq 2$, we have again $|\pi_{L_1}(w)| \leq k-1$.

Consider any vertex $w \in \mathcal{I}_2$. First assume that $v \neq y$. Then $L(w) < L(y)$ and $L(w) < L(z)$ and so there is a vertex from p'' and another one from p'_v such that both those vertices are in $\pi_L(w)$ and not in $\pi_{L_1}(w)$. Since $|\pi_L(w)| \leq k$, it follows that $|\pi_{L_1}(w)| \leq k-2$. Now assume that $v = y$. But y is either the rightmost or the second rightmost vertex of L, so $\mathcal{I}_2$ must have only one vertex, so $w = v$, and so it is the case that $w = v = \mathrm{right}(v)$. If v is the rightmost vertex of L (and of L_1) its separation is zero so it is true that $|\pi_{L_1}(w)| \leq k-2$, k being at least two. If v is the second rightmost vertex of L then v is still the rightmost vertex in L_1 and its separation in L_1 is zero, so again we have $|\pi_{L_1}(w)| \leq k-2$.

Conclude that the separation of any vertex under L_1 is equal to $k-1$ minus the number of intervals among $\mathcal{I}_1$ and $\mathcal{I}_2$ that it is in. By Definition 6, L_1 is $(k-1)$-stretchable with respect to u and v so that u is associated with the left

direction and v, with the right one. So, L_1 is $(k-1)$-affixable with respect to u and v.

Part I.2.   Assume that $L(v) < L(u)$, thus $L_1(v) < L_1(u)$, and let $\mathcal{I}_1$ and $\mathcal{I}_2$ have the same meaning as above. L_1 looks like:

$$L_1 = \underbrace{\ldots\ldots\ldots\ldots,v,\ldots\ldots\ldots}_{\text{interval } \mathcal{I}_1},u,\ldots\ldots\ldots,\underbrace{\text{right}(v),\ldots\ldots\ldots\ldots}_{\text{interval } \mathcal{I}_2}$$

Note that $\mathcal{I}_1$ and $\mathcal{I}_2$ cannot overlap because u and v are adjacent and so right(v) cannot be left of u. However, they can cover the whole layout in case that right(v) is u. The proof is the same as above, the only difference being that $\mathcal{I}_1$ now cannot be empty because v is in it so that possiblity is not considered.

Part II.   Assume that $\mathcal{F}$ is not in $\mathcal{P}$. Then $\mathcal{F}$ is a face in some child $\widetilde{G}$ of $\mathcal{P}$. We know from Part I of this proof that $\widetilde{G}$ is $(k-1)$-affixable. Apply Lemma 7 to $\widetilde{G}$ and conclude both children of $\mathcal{F}$ are $(k-1)$-affixable. $\square$

5. Conclusion. Theorem 2 is the first step towards the development of a fast, practical algorithm for the vertex separation of maximal outerplanar graphs. This seems to be a task considerably harder and more involved than the development of a fast, pratical algorithm for the VS of trees ([7]). There is anecdotal evidence ([12]) that at a certain time before 2000 substantial effort had been thrown at developing such an algorithm with no results. Theorem 2 is a solid theoretical foundation for such an algorithm, which, at that moment at least, is only hypothetical. Note that the algorithm for the VS of trees rests upon Theorem 3.1 from [7] that reduces the problem of the VS of the whole tree to the VS of subtrees. Likewise, our Theorem 2 reduces the problem of the VS of a mop to the problem of computing a certain parameter of submops. However, in the case of mops, the parameter we reduce to is not VS but what we defined as affixability. Clearly, if we are to develop the said algorithm, we have to find a similar reduction of the affixability of a (rooted) mop to certain parameters of subgraphs of its. That turned out to be a formidable challenge, algthough we achieved certain results in that direction: for instance, it turned out that the affixability of a mop G being $\leq k$ may depend with an "if and only if" dependence on whether the VS of a subgraph of G that is not a mop but a 1-connected sequence of mops (exactly as G_1 and G_2 from Figure 7 on page 216) is $\leq k-3$. It follows that the problem for the VS of mops contains as a subproblem the problem for the VS of a class of graphs that is more general than mops. In other words, the solution of VS (meaning an if and only if condition for

the VS of a mop being at most k) is no easier than the solution for the VS of a class of 1-connected graphs whose blocks are mops.

REFERENCES

[1] BODLAENDER H. L., F. V. FOMIN. Approximation of pathwidth of outer-planar graphs. Technical Report UU-CS-2000-23, Department of Computer Science, Utrecht University, Utrecht, the Netherlands, 2000.

[2] BODLAENDER H. L., T. KLOKS. Efficient and constructive algorithms for the pathwidth and treewidth of graphs. *Journal of Algorithms*, **21**, No 2 (September 1996), 358–402.

[3] COUDERT D., F. HUC, J.-S. SERENI. Pathwidth of outerplanar graphs. *Journal of Graph Theory*, **55**, No 1 (2007), 27–41, `http://kam.mff.cuni.cz/~sereni/Articles/CHS07.pdf`, 2008.

[4] DOWNEY R. G., M. R. FELLOWS. Parameterized Complexity. Springer, 1999.

[5] DESSMARK A., A. LINGAS, A. PROSKUROWSKI. Faster algorithms for sub-graph isomorphism of k-connected partial k-trees. *Algorithmica*, **27**, No 3 (2000), 337–347, `http://www.cs.uoregon.edu/~andrzej/pubs/algorithmica99.ps`, 2008.

[6] ELLIS J., M. MARKOV. Computing the vertex separation of unicyclic graphs. *Information and Computation*, **192** (2004), 123–161, `http://www.cs.uvic.ca/~jellis/Publications/unicyclic.ps`, 2008.

[7] ELLIS J. A., I. H. SUDBOROUGH, J. S. TURNER. The vertex separation and search number of a graph. *Information and Computation*, **113**, No 1 (15 August 1994), 50–79.

[8] GOVINDAN R., M. A. LANGSTON, X. YAN. Approximating the pathwidth of outerplanar graphs. *Information Processing Letters*, **68**, No 1 (15 October 1998), 17–23.

[9] KLOKS T. Treewidth: Computations and approximations. *Lecture Notes in Computer Science*, vol. **842**, 1994, ix + 209 pp.

[10] Markov M. A fast practical algorithm for the vertex separation of un-
     cyclic graphs. Master's thesis, University of Victoria, December 2004,
     `http://minko.m.markov.googlepages.com/diss.pdf`, 2008.

[11] Markov M. The reversibility of graph layout multistretchability. In: New
     Trends in Mathematics and Informatics, 2007,
     `http://minko.m.markov.googlepages.com/reversibility.pdf`, 2008.

[12] Proskurowski A., Personal communications, April 2000.

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