Serdica J. Computing **1** (2007), 45–72

Serdica
Journal of Computing

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON THE VERTEX SEPARATION OF CACTUS GRAPHS

Minko Markov

ABSTRACT. This paper is part of a work in progress whose goal is to construct a fast, practical algorithm for the vertex separation (VS) of cactus graphs. We prove a "main theorem for cacti", a necessary and sufficient condition for the VS of a cactus graph being k. Further, we investigate the ensuing ramifications that prevent the construction of an algorithm based on that theorem only.

1. Introduction.

VERTEX SEPARATION is an NP-complete problem on undirected graphs with numerous practical applications in diverse areas such as natural language processing, VLSI design, network reliability, computational biology and others. It is equivalent to other, seemingly unrelated, graph problems such as NODE SEARCH NUMBER, INTERVAL THICKNESS and the famous PATH-WIDTH. For a brief survey of some theoretical and practical applications of VERTEX SEPARATION and its relation to other problems, see [6].

The NP-completeness of a computational problem is a strong evidence that it is intractable in general. There are polynomial time algorithms for

ACM Computing Classification System (1998): G.2.2.

Key words: algorithmic graph theory, computational complexity, vertex separation, linear layout, layout extensibility, layout stretchability, cactus graph

Vertex Separation on classes of graphs. However, often those algorithms have only theoretical importance because the polynomial upper bound on the running time has a high degree, *e.g.* the algorithm of Bodlaender and Kloks on graphs of treewidth k runs in time $\Omega(n^{4k+3})$ ([1]), and because they are extremely difficult to implement in practice.

In the context of partial k-trees with fixed k, till recently the only fast, practical algorithm was the linear-time algorithm for Vertex Separation on trees (*i.e.* 1-trees) due to Ellis, Sudborough, and Turner [3]. There is anecdotal evidence of other researchers trying unsuccessfully to discover a fast, practical algorithm for the VS of unicyclic graphs, a unicyclic graph being a tree plus an extra edge. Such an algorithm is discovered by Ellis and Markov in [2] and although it lacks the elegance of the algorithm on trees it is still a first step in dealing with the VS of graphs beyond trees.

Cactus graphs are a natural extension of unicyclic graphs since a cactus graph is, informally, a collection of unicyclic graphs, joined together in a tree-like way. Therefore, it is natural to try developing an algorithm for the VS of cactus graphs, having the knowledge how to deal with VS on unicyclic graphs.

We approach VS on cactus graphs in a way similar to the approach of [3] on trees: discover and verify a so called "main theorem" for the VS of the class of graphs under consideration, that is, an if-and-only-if condition for the VS of a graph from that class being k, the condition being in terms of the VS of subgraphs. The "main theorem" in [3, Theorem 3.1] says that the VS of a tree is k if and only if k is the smallest number, such that for any vertex of that tree, the deletion of that vertex leaves us with at most two subtrees of VS $= k$. Having proved the theorem, Ellis, Sudborough, and Turner derive an elegant, linear-time, divide-and-conquer algorithm based solely on it and the obvious fact that the VS of a single vertex is zero. That approach is quite different from the *ad hoc* approach of [2] on unicyclic graphs that boils down to running the algorithm from [3] on the trees around the cycle and then considering a complicated case-subcase hierarchy of possibilities.

The work on unicyclic graphs has one important and valuable concept, however: the idea of k-*conforming layout* (see [2, Definition 3.3]), which is renamed to k-*stretchable layout* in [6, Definition 5]. In the current paper, we stick to the term "stretchable". That concept, we believe, is crucial for the solution of VS on cacti just as it is crucial for the solution of VS on unicyclic graphs.

The "main theorem" for the VS of cactus graphs that we propose here reduces the VS of the whole cactus to both the VS *and* the stretchability of subgraphs of its. It is not surprising that theorem does not lead immediately to an algorithm (in contrast to the "main theorem" for trees) because of the reduction of one parameter, the vertex separation, to another one, the stretchability. It is obvious that we need more theoretical results that clarify how the stretchability is related to the VS and stretchabilities of subgraphs.

That ramification, namely the introduction of a second parameter in the main theorem, seems unavoidable and inherent in the nature of cactus graphs. The knowledge accumulated in the solution of VS on unicyclic graphs, namely the way we deal with stretchability of unicyclic graphs, is very helpful when one begins to tackle stretchability of cactus graphs. Unfortunately, one quickly discovers that stretchability with respect to a single pair of vertices from some cycle—and that is what we call simply "stretchability" in the current paper—can in some cases be reduced to stretchability with respect to *two* pairs of vertices from two cycles, which in its turn can be reduced to stretchability with respect to *three* pairs of vertices from three cycles, and so on. In the case of unicyclic graphs that ramification does not show up simply because they have only one cycle.

Having discovered that, it seems futile to try to develop an algorithm for the VS of cactus graphs based on the said main theorem: dealing with the stretchability with respect to a single pair of vertices in unicyclic graphs is rather complicated and "convoluted" in the sense of the case-subcase hierarchy of possibilities that has to be considered, so one imagines that dealing with stretchability with respect to two pairs of vertices would be much more complicated. And that pales when one considers there is no *a priori* fixed bound on the number of vertex pairs we have to deal with: for any positive integer m, the stretchability with respect to m vertex pairs from m cycles may be reduced to stretchablity with respect to $m + 1$ vertex pairs from $m + 1$ cycles.

We believe we found a way out of the "morass of growing numbers of vertex pairs we have to stretch relative to" but that is still work in progress, very extensive at that. In this paper we only prove the main theorem for cactus graphs and we show that the stretchability with respect to one pair of vertices may reduce to the stretchability with respect to two pairs of vertices, *etc.* We believe that the mentioned way out—the real solution of VS on cactus graphs—uses a generalisation of the current main theorem, so the results of this paper are an important first step towards constructing a fast, practical algorithm for the VS of cactus graphs.

2. Background.

2.1. Basic definitions. We assume the reader is familiar with the basic definitions of Graph Theory (see, for example, [4]). Our standard notation for a graph is "$G = (V, E)$", where V is the vertex set and E is the edge set. The graphs that we consider are undirected graphs without multiple edges or loops. If G is a graph and we write "$u \in G$", we mean that u is a vertex in G. For any vertex v that is adjacent to a vertex u, we say that u and v are *neighbours.* If $p = u_1, u_2, \ldots, u_k$ is a path, we say that u_1 and u_k are *the endpoints of* p. The notation "$u_1 \overset{p}{\frown} u_k$" is an abbreviation for "there is a path p with endpoints u_1 and u_k".

If $= (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs, by "$G_1 \cap G_2$" we denote $V_1 \cap V_2$. Furthermore, if $V_1 \cap V_2 = \{u\}$ for some $u \in V_1, V_2$, we write "$G_1 \cap G_2 = u$".

To *delete* a vertex u from $G = (V, E)$ means to transform G into $(V \setminus \{u\}, E \setminus E_u)$, where E_u is the set of edges in G incident with u. To *remove* an edge e from $G = (V, E)$ means to tranform G into $(V, E \setminus \{e\})$. To delete a subgraph G_1 from G means to delete all the vertices of G_1 from G.

Suppose that G_1 and G_2 are distinct graphs both having a vertex named x and that all the other vertices in G_1 and G_2 have distinct names. Then "$G_1 \oplus G_2$" denotes the graph, obtained from G_1 and G_2 by identifying x in one graph with x in the other one. Formally, that means: suppose that u_1, u_2, $\ldots$, u_k are all the neighbours of x in G_1 and v_1, v_2, $\ldots$, v_t are all the neighbours of x in G_2; delete x from G_1 and x from G_2, add a new vertex z, connect z to each of u_1, u_2, $\ldots$, u_k, v_1, v_2, $\ldots$, v_t, by an edge, and rename z to x. The new graph is said to be obtained by *welding* G_1 *and* G_2. We generalise the welding operation to arbitrarily many graphs G_1, G_2, $\ldots$, G_t as follows. Suppose that all the vertices in them have distinct names, except that G_i and G_{i+1} have a common name vertex x_i, for $1 \le i \le t-1$. Then *the welding of* G_1, G_2, $\ldots$, G_t is the graph $((G_1 \oplus G_2) \ldots) \oplus G_t$. The welding operation is clearly associative so we write simply "$G_1 \oplus G_2 \oplus \ldots \oplus G_t$".

A linear layout of a graph $G = (V, E)$, or simply *a layout*, is a bijective function $L: V \to \{1, 2, \ldots, |V|\}$. For any layout L and vertex u, $\pi_L(u)$ is defined to be $\pi_L(u) = \{v \in V \,|\, L(v) \le L(u)$, and for some $w \in V$, $L(w) > L(u)$ and $(v, w) \in E\}$. *The separation of* u *under* L is $|\pi_L(u)|$. For any $v \in \pi_L(u)$, we say that v *contributes* to the separation of u. *The vertex separation of* G *under* L is $vs_L(G) = \max_{u \in V}(\,|\pi_L(u)|\,)$ and *the vertex separation of* G is $vs(G) = \min\{\, vs_L(G) \,|\, L$ is a linear layout of $G\}$. Any layout L such that $vs_L(G) = vs(G)$ is called *optimal.* If for some $u \in G$, $|\pi_L(u)| = vs_L(G)$, we call u *heavy under L*, or simply *heavy* in case that it is clear which layout we mean. If p is a path in G, we say that p *contributes to the separation of* u *under* L if at least one vertex from p is in $\pi_L(u)$.

For any $u, v \in G$ such that $L(u) < L(v)$, we say that under L, u is *left of* v, and v is *right of* u. Let w be the rightmost neighbour of u, under L. The vertex *right*(u) is defined as follows: if $L(w) > L(u)$, then right(u) is w, and otherwise right(u) is u.

Throughout this work, we think of linear layouts as of lists of vertices, rather than as mappings of vertices on integers. If $G_1 = (V_1, E_1)$ is a proper subgraph of G, then *the linear sublayout*, or simply *the sublayout*, of G_1 under L, is the ordering of the vertices from V_1 under L. Further, for a vertex $u \in V$, we denote by "$L - u$" the list L with u deleted from it, and the other vertices left in the same relative order. If $L = u_1, u_2, \ldots, u_n$ is a layout, then *an interval in* L is a contiguous, possibly empty subsequence $u_i, u_{i+1}, \ldots, u_{i+j}$ of L.

2.2. Extensibility of a layout and a graph. The following two definitions are from [2] (see Definition 2.1 on page 126). Prior to [2], Skodinis [7] introduces similar concepts. Suppose that $G = (V, E)$ is a graph, L is a layout of G, and k is a positive integer.

Definition 1 (left-extensible layout)**.** *For any* $u \in V$*, we say that* L *is* left-extensible with respect to k and u*, if all vertices left of* u *have separation strictly less than* k*, and the remaining vertices have separation less than or equal to* k*. We denote that by "*L *is* (k)*-left*(u)*-ext". If* $k = vs_L(G)$*, we say that* L *is* left-extensible with respect to u*, denoted by "*L *is left*(u)*-ext".*

Observation 1. *Suppose that* $G = (V, E)$ *is a graph and* L *is a layout of* G*. If* $L(u) = 1$ *and* $vs_L(G) \leq k$*, then* L *is* (k)*-left*(u)*-ext. If* L *is* (k)*-left*(u)*-ext and* L *is modified by placing* u *at the leftmost position and keeping the relative order of the other vertices, the modified layout is still* (k)*-left*(u)*-ext.*

The second claim is evident, but, if in doubt, consider the vertices to the left of u in the original L, and observe that their separation may indeed increase by at most one after u is moved, but the resulting layout still conforms to Definition 1. Having in mind Observation 1 and the obvious fact that if $L(u) = 1$ then L is (k)-left(u)-ext, we can assume without loss of generality that for any left-extensible with respect to u and k layout L, u is the leftmost vertex.

Definition 2 (right-extensible layout)**.** *For any* $u \in V$*, we say that* L *is* right-extensible with respect to k and u*, if all vertices right of and including right*(u) *are of separation strictly less than* k*, and the other vertices are of separation less than or equal to* k*. That is denoted by "*L *is* (k)*-right*(u)*-ext". If* $k = vs_L(G)$*, we say that* L *is* right-extensible with respect to u*, denoted by "*L *is right*(u)*-ext".*

Lemma 3.3 and Lemma 3.4 of [2] prove that there exists a (k)-left(u)-ext layout of G if and only if there exists a (k)-right(u)-ext layout of G. In other words, the extensibility property with respect to a vertex is reversible. Therefore, we can make the following definition.

Definition 3. *If* L *is* (k)*-left*(u)*-ext or* (k)*-right*(u)*-ext, we say that* L is k-extensible with repsect to u, *denoted by "*L *is* (k)-(u)*-ext". When* $k = vs_L(G)$*, we say that* L is extensible with respect to u, *denoted by "*L *is* (u)*-ext". When* k *is the minimum number such that* L *is* (k)-(u)*-ext, we write "*L *is* $\sharp(k)$-(u)*-ext". When* L *is not* (k)-(u)*-ext, we write "*L *is* $\neg(k)$-(u)*-ext".*

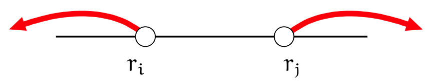
2.3. The usefulness of the concept of extensibility. Now we explain informally why extensibility is useful and the rationale for choosing this word. Think of layouts not as of mappings or vertex sequences but as of actual drawings of graphs in the plane with the vertices arranged along a line and the edges being drawn between them, possibly intersecting. Suppose that L is a layout for a graph G such that $vs_L(G) = k$. We want to use L in order to construct a bigger layout, say L', in which L will be a sublayout. In other words, G is a subgraph of some graph G' and, using L as a building block, we wish to construct a layout for G'. Suppose that u is a vertex in G and the only connecting edge e between G and the remainder of G' is incident with u. Suppose we have some layout L'' for the remainder of G and construct L' by placing L'' to the left of L. Our goal is to keep the overall separation as small as possible so, as far as L is concerned, we want not to increase the separation of any vertex in L beyond k. However, the placement of e may increment by one the separation of some vertices of L, namely, the vertices that are to the left of u. Obviously, the separation of L after e is placed will not exceed k if and only if L is (k)-left(u)-ext. Likewise for the right-extensibility.

So, that is the importance of the extensibility concept: it allows us to construct bigger layouts out of smaller layouts using connections to either the left or the right while keeping the overall separation no bigger than that of the smaller layouts. The name comes from the fact that L is *extended* by an edge that is not in G.

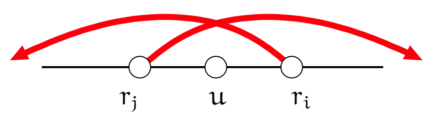
2.4. Stretchability of a layout and a graph. The concept of stretchability is crucial for the solution of VERTEX SEPARATION on cacti. The solution of VERTEX SEPARATION on unicyclic graphs, which are one-cycle cacti, uses stretchability, too – [2] introduces the term "k-conforming layout" with the same intended meaning as "k-stretchable layout" in this work. Namely, that a layout can be *simultaneously* extended both to the left and to the right, while the separation of all vertices layout stays at most k.

Definition 4 ([2], Definition 3.3 on page 140)**.** *Let* U *be a unicyclic graph and let* r_i *and* r_j *be cycle vertices. A layout* L *for* U *will be said to be* k-conforming *with respect to* r_i *and* r_j *if either* $vs_L(U) = k$ *and* L *is left extensible with respect to* r_i *and* k*, and right extensible with respect to* r_j *and* k*, or* $vs_L(U) \leq k - 1$*.*

Unfortunately, there is an "error" in this definition, in the sense that it does not necessarily provide the intended meaning as stated above when the separation of the graph under L is k. Suppose that the separation under L is k and L is left extensible with respect to r_i and k and L is right-extensible with respect to r_j and k. We want to extend simultaneously from r_i in the left direction and from r_i in the right direction. If $\mathsf{L}(r_i) < \mathsf{L}(r_j)$ (typically, one would imagine r_i is left of r_j when r_i is associated with the left direction and r_j with the right one), the separation after the two extendings does not exceed k, so the definition "works":



However, when $\mathsf{L}(r_j) < \mathsf{L}(r_i)$, the simultaneous extending from r_i to the left and from r_j to the right can cause the separation of some vertex u that is between r_j and r_i to become $k+1$:



To see that is true, imagine the separation of u is $k-1$ before the extendings and u is to the right of, or coinciding with, right(r_j).

So, indeed, L being left-extensible with respect to r_i and k *and* right-extensible with respect to r_j and k does not guarantee that the separation stays at most k if we extend simultaneously from r_i to the left and from r_j to the right. We remedy the situation by using the following definition instead, which is from [6].

Definition 5 ([6], Definition 5 on page 19: stretchable layout with respect to two vertices)**.** *Let* u *and* v *be two vertices from* L*, not necessarily distinct. Let* $\mathcal{I}_u$ *be the possibly empty interval* $\left[\mathsf{L}^{-1}(1), \ldots, \mathsf{L}^{-1}(\mathsf{L}(u)-1)\right]$ *and* $\mathcal{I}_v$ *be the non-empty interval* $\left[\mathit{right}(v), \ldots, \mathsf{L}^{-1}(|\mathsf{V}|)\right]$*. Let* $\mathcal{J}_v$ *be the possibly empty interval* $\left[\mathsf{L}^{-1}(1), \ldots, \mathsf{L}^{-1}(\mathsf{L}(v)-1)\right]$ *and* $\mathcal{J}_u$ *be the non-empty interval* $\left[\mathit{right}(u), \ldots, \mathsf{L}^{-1}(|\mathsf{V}|)\right]$*. We say that* L is k-stretchable with respect to u and v *if at least one of the following holds:*

- *the separation of any vertex in* L *is at most* k *minus the number of intervals from* $\mathcal{I}_u, \mathcal{I}_v$ *that it is in;*

- *the separation of any vertex in* L *is at most* k *minus the number of intervals from* $\mathcal{J}_u, \mathcal{J}_v$ *that it is in.*

In the former case, we say also that u is associated with the left direction and v with the right direction, *and in the latter case we say the opposite.*

Theorem 2 from [6] proves that there is a k-stretchable with respect to u and v layout where u is associated with the left direction and v is associated with the right direction, if and only if there exists a k-stretchable with respect to u and v layout where v is associated with the left direction and u, with the right direction. Therefore, when we talk about a k-stretchable graph with respect to two vertices, we do not associate the vertices with directions, since there exist layouts for either case.

Having in mind Observation 1, we assume without loss of generality that for any layout L that is k-stretchable with respect to u and v, and u is associated with the left direction, $L(u) = 1$ and L is (k)-right(v)-ext. The fact that L is k-stretchable with respect to u and v is denoted by "L is (k)-(u, v)-stretchable".

The definition of stretchability does not require that the two vertices u and v are distinct. In case that the vertices coincide, we make the following observation.

Observation 2. *Suppose that* G *is a graph,* u *is a vertex in it, and* L *is a layout for* G*.* L *being* (k)-(u, u)-*stretchable is equivalent to* $L - u$ *being of separation* $\leq k - 1$*.*

If G has a layout L that is stretchable in a certain way, we say that G is stretchable in that way, too. When we say that G is not stretchable in a certain way, we mean that there is no layout of G, stretchable in this way.

2.5. Definitions on cacti.

Definition 6 (cactus graph)**.**

- *A tree is a cactus graph.*
- *If* G_1*,* G_2*, ...,* G_k *are cactus graphs for some* k *such that* $k \geq 3$*, and for* $1 \leq i \leq k$*,* u_i *is a vertex in* G_i*, then the graph, obtained from* G_1*,* G_2*, ...,* G_k *by adding the edges* (u_1, u_2)*,* (u_2, u_3)*, ...,* (u_{k-1}, u_k)*,* (u_1, u_k)*, is a cactus graph.*
- *Nothing else is a cactus graph.*

It is easy to see this definition is equivalent to the conventional one: a cactus graph is a connected graph whose blocks are either cycles or single edges. In this work, we use the short name *cactus*, plural *cacti*.

Let $G = (V, E)$ be a cactus. An edge $e \in E$ is *a cycle edge* if it is in some cycle of G, and *tree edge*, otherwise. *A subcactus of* G is a subgraph of G that is a cactus. If u is a vertex in G then $G - u$ is a collection of subcacti G_1, G_2, ..., G_t and we say that u *begets* G_1, G_2, ..., G_t.

Definition 7 (constituents of a cycle). *Let* G *be a cactus and* s *be a cycle in* G *that has* k *vertices. If we remove the edges of* s*, we obtain* k *connected components that are called* the constituents of s. *For each vertex* $u \in s$*, the* u*-constituent of* s *is the constituent that* u *belongs to.*

Let u *be a vertex in* s. *We say that we* collapse the u-constituent of s *when we delete all the vertices of the* u*-constituent except* u. *The result of the collapsing is denoted by "*$G \ominus u$*" when* s *is understood and by "*$G[s] \ominus u$*" otherwise.*

For any two vertices u, v *from* s*, adjacent or not, if we collapse both the* u*- and the* v*-constituent, we denote that by "*$G \ominus [u, v]$*" when* s *is understood and by "*$G[s] \ominus [u, v]$[1]*" otherwise.*

Clearly, if u is a vertex in several cycles $s_1 \ldots s_t$, all $G[s_i] \ominus u$ are distinct graphs. We emphasise that u is vertex of degree two in $G \ominus u$ and u, v are vertices of degree two in $G \ominus [u, v]$.

Definition 8 (archs of a cycle). *Let* G *be a cactus and* s *be a cycle in it. Let* $v_1, v_2, \ldots, v_t$ *be a proper subset of the vertex set of* s*, for some* $t \geq 1$. *If we delete every* v_i*-constituent of* s*, for* $1 \leq i \leq t$*, we obtain* k *connected components for some* k*,* $1 \leq k \leq t$. *These* k *connected components are called* the archs of s relative to $v_1, v_2, \ldots, v_t$.

Definition 9 (rooted cactus). *Any cactus becomes* a rooted cactus *when we choose one vertex in it to be* the root vertex, *or simply* the root. *Assume that* u *is the root of* G. *The subcacti begotten by* u *are* the children of u. *The children of* u *are partitioned into* tree children *and* cycle children. *The former ones are those children that are connected to* u *by a (single) tree edge, and the latter ones are those children that are connected to* u *by (two) cycle edges. For each cycle child, its* parental cycle *is the cycle in* G *that contains the two said cycle edges. See Figure 1 for illustration.*

Any tree child of the root is in its turn a rooted cactus. If G_i *is a tree child of* u*, its root is* v_i*, where* (u, v_i) *is the tree edge connecting* u *to* G_i*, as*

[1] The notation "$G[s] \ominus [u, v]$" contains redundancy since there can be at most one cycle containing two vertices and therefore u and v determine s uniquely, but we find it convenient to remind which cycle we have in mind.

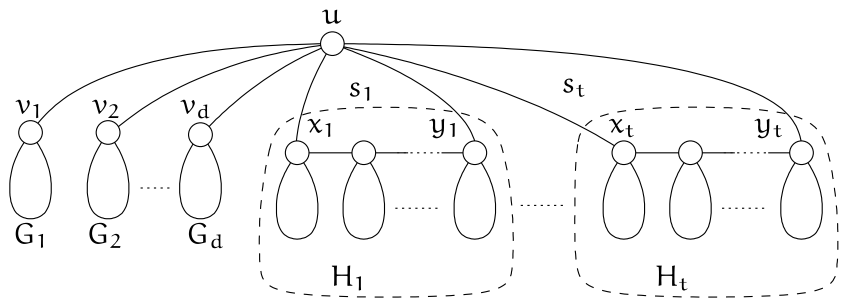


Fig. 1. The cactus is rooted at u. The children of u are G_1, G_2, …, G_d, H_1, …, H_t. Further, G_1, G_2, …, G_d are the tree children, and H_1, …, H_t are the cycle children. Each G_i is rooted at the respective v_i. For each H_i, s_i is its parental cycle, and each constituent of s_i that is in H_i is a rooted cactus with root vertex the vertex from s_i that is in it

shown on Figure 1. The cycle children of u *are* not *considered to be rooted. However, assuming that* H_i *is a cycle child of* u*,* (u, x_i) *and* (u, y_i) *are the two cycle edges connecting* u *to* H_i*, and* s_i *is the cycle containing* (u, x_i) *and* (u, y_i) *(see Figure 1), we say that* s_i *is the* parental cycle of H_i*, and every constituent of* s_i *except for the constituent containing* u *is considered to be a rooted cactus with root vertex the corresponding vertex from* s_i*. That is, the cycle children of* u *are not rooted cacti but each one of them is a collection of rooted cacti whose roots lie on a path.*

For any cycle child of u*, we say that* u *is the root of its parental cycle. For any child, be it a tree or cycle child,* G' *of* u*, by "*$G' + u$*" we denote the graph that consists of* G' *plus vertex* u *plus the one or two edges that connect* G' *to* u*. If* G' *is a cycle child with parental cycle* s*, then "*$G[u|s]$*" is an alternative notation for* $G' + u$*.*

Clearly, if G is a rooted cactus with root u, the number of children of u is equal to the number of subcacti begotten by u. For any other vertex v in G, the number of children of v is equal to the number of subcacti begotten by v minus one.

Definition 10 (c-path)**.** *Suppose that* G *is a cactus,* u *and* v *are vertices in it, and* p *is any path in* G *with endpoints* u *and* v*. Think of* p *as an alternating sequence of vertices and edges. For any maximal subpath* $q = w_1, e_1, w_2, e_2, \ldots, e_{t-1}, w_t$ *of* p *where* $t \geq 2$*,* $w_1, \ldots, w_t$ *are vertices, and* $e_1, \ldots, e_{t-1}$ *are edges, such that* $e_1, \ldots, e_{t-1}$ *are cycle edges from some cycle* s*, substitute* q *with* w_1, s, w_t*. The obtained sequence* C *of vertices and cycles is called* a c-path.

We say that C connects u and v*, and that* u *and* v *are* the endpoints *of*

C. *Any vertex in a c-path that is not an endpoint is* an internal vertex. *The said vertices* w_1 *and* w_t *are* the attachment vertices of s in C. *The vertices and cycles in* C *are* the elements of C. *The maximal vertex subsequences in* C *are* the paths in C.

It follows from Definition 10 that any vertex in C, be it an attachment vertex or not, can possibly be a vertex in arbitrarily many cycles from G that are not elements of C, and that in any c-path, there is at least one vertex between any two cycles – so, we can think of a c-path as of an alternating sequence of paths and cycles, the paths being one more than the cycles.

It is easy to see that there is a unique c-path C that connects any two vertices in a cactus. Thus, c-paths in cacti are analogous to paths in trees: in a tree there is a unique path between any two vertices, in a cactus there is a unique c-path between any two vertices.

Suppose that C is a c-path with one endvertex u. By "$C - u$" we mean the sequence, obtained from C by deleting u. Note that the sequence $C - u$ may have a cycle at one end, *i.e.* it may not be a c-path. However, we use that notation only in the following context: two c-paths, say C_1 and C_2, have a common endpoint u and are disjoint except for u. Define that $C_3 = C_2 - u$. Then "C_1, C_3" denotes a c-path.

For any two vertices u and v in a c-path C, we say that u and v are *c-path neighbours in* C if either u and v occur in C next to each other, or there is a single cycle between them.

Definition 11 (children of a vertex in a c-path)**.** *Suppose that* G *is a cactus and* C *is a c-path in it. Suppose that* a *and* z *are the endpoints of* C*. For each vertex* u *in* C*,* the children of u in C *are those subcacti begotten by* u *that contain neither* a *nor* z*.*

Note that any path is a c-path as well, so we can define children of a vertex in a path using Definition 11.

Definition 12. *Suppose that* G *is a cactus and* C *is a c-path in it. For any cycle* s *in* C*,* the s-constituent of C *is* $G[s] \ominus [u, v]$*, where* u *and* v *are the attachment vertices of* s *in* C*.*

A convention we stick to is that if we discuss the stretchability of the s-constituent of C and do not specify the vertices with respect to which that stretchability is, we invariably have in mind stretchability with respect to the vertices of attachment of s in C.

Definition 13. *Suppose that* G *is a cactus and* C *is a c-path in it. For any path* $p = u, v, \ldots, w$ *in* C*,* the p-constituent of C *is the subgraph of* G*,* *induced by the vertex set that consists of* u*,* v*, ...,* w*, plus the vertices from all the children of* u*,* v*, ...,* w*, in* C*.*

Definition 14. *Suppose that* G *is a cactus and* s_1*,* s_2 *are two not necessarily vertex-disjoint cycles in* G*. Clearly, there is a unique vertex* $u_1 \in s_1$ *and there is a unique vertex* $u_2 \in s_2$*, such that the c-path* C *connecting* u_1 *and* u_2 *contains neither* s_1 *nor* s_2*. We call* C*,* the c-path connecting s_1 and s_2.

Note that C in Definition 14 can be as small as a single vertex—in case that s_1 and s_2 are not vertex-disjoint.

Definition 15 (favourable vertex and favourable vertices)**.** *Suppose that* G *is a cactus and* s *is a cycle in it. For any two not necessarily distinct vertices* $u, v \in s$*,* u *and* v *are* k-favourable in s *if* $G[s] \ominus [u, v]$ *is* k*-stretchable. If* u *and* v *are distinct, we say that* u *and* v *are a* k-favourable pair. *A vertex* u *from* s *is* k*-favourable in* s *if there exists a vertex* w *from* s*, possibly coinciding with* u*, such that* u *and* w *are* k*-favourable. When a vertex or two vertices from* s *are not* k*-favourable, we use the term* k-unfavourable.

Recall that our definition of stretchability with respect to vertices u and v allows the possibility that $u = v$. By Observation 2 on page 80, $G[s] \ominus [u, u]$ being k-stretchable is the same thing as the arch of s relative to u being of separation $< k$. So, if u is k-unfavourable then the arch of s relative to u is of separation $\geq k$. However, the fact that the arch of s relative to u is of separation $\geq k$ does not imply that u is k-unfavourable because there can exist a vertex $x \in s$, $x \neq u$, such that $G[s] \ominus [u, x]$ is k-stretchable. To see that the latter is true, imagine that the x-constituent of s has separation precisely k and the other constituents have separations $\leq k - 2$.

Definition 16 (important vertex and important cycle)**.** *Suppose that* G *is a non-rooted cactus. A vertex in* G *is* k-important *if it begets precisely two subcacti of separation* k *and all the other begotten subcacti are of separation less than* k*. A cycle* s *is* k*-important if for every vertex* $u \in s$*, the arch of* s *relative to* u *has separation* k*. When the number* k *is understood, we say simply* important*.*

Definition 17 (criticality)**.** *Suppose that* G *is a rooted cactus and* $vs(G) = k$*. For any vertex* $u \in G$ *such that precisely two children of* u *have separation* k*, we say that* u *is* a k-critical vertex*, or simply* a critical vertex *when* k *is understood. For any cycle* s *in* G *such that the root of* s *is not* k*-favourable we say that* s *is* a k-critical cycle*, or simply* a critical cycle *when* k *is understood. If* $vs(G) = k$ *and* G *has a critical vertex or cycle then* G *itself is called critical. Otherwise,* G *is* noncritical.

Given that $\mathsf{vs}(G) = k$, by Theorem 1 no vertex in G can have more than two separation k children, therefore a vertex v that is not k-critical may have at most one separation k child.

Definition 18 (k-compliant c-path)**.** *Suppose that* G *is a cactus. A c-path* C *in* G *is called* k-compliant *if for each vertex* u *in* C*, the children of* u *in* C *are of separation less than* k*, and for each cycle* s *in* C*, the attachment vertices of* s *in* C *are* k*-favourable in* s*.*

3. Methods for construction of layouts and c-paths.

Method 1. *We are given a cactus* G*,* u *is a vertex in it, and* G_1*,* G_2*, ...,* G_t *are the subcacti begotten by* u*. The input is a multitude of layouts* L_i *for* G_i*, for* $1 \le i \le t$*, such that* $\mathsf{vs}_{L_i}(G_i) < k$*. The output is a layout* L *for* G *such that* L *is* (k)*-*(u, u)*-stretchable.*

Construction. $L = u, L_1, L_2, \ldots, L_t$. It is immediately obvious that L is indeed (k)-(u, u)-stretchable. □

Method 2. *We are given a cactus* G*,* $p = u_1, u_2, \ldots, u_q$ *is a path in it such that* p *has no cycle edges, and* G^i_1*,* G^i_2*, ...,* $G^i_{t_i}$ *are the children of* u_i *in* p*, for* $1 \le i \le q$*. The input is a set of layouts* L^i_j *for* G^i_j *for* $1 \le i \le q$ *and* $1 \le j \le t_i$*, each of those layouts being of separation at most* $k-1$*. The output is a layout* L *for* G *such that* L *is* (k)*-*(u_1, u_q)*-stretchable.*

Construction. Note that if p has no cycles edges then p is a c-path as well, so Definition 11 holds. Using Method 1, build a (k)-(u_i, u_i)-stretchable layout L_i for G_i, where, for $1 \le i \le q$, G_i is the subgraph of G induced by the union of the vertices of G^i_1, G^i_2, ..., $G^i_{t_i}$, and u_i. The desired output is $L = L_1$, $L_2, \ldots, L_q$. The fact that L is (k)-(u_1, u_q)-stretchable follows immediately from the definition of stretchable, applied to each L_i in L. □

Method 3. *We are given a cactus* G *and a* k*-compliant c-path* $C = p_1$*,* $s_1, p_2, s_2, \ldots, s_{t-1}, p_t$ *in it where* $p_1 = u^1_1, u^1_2, \ldots, u^1_{i_1}$*,* $p_2 = u^2_1, u^2_2, \ldots, u^2_{i_2}$*, ...,* $p_t = u^t_1, u^t_2, \ldots, u^t_{i_t}$ *are the paths in* C*. Clearly, the attachment vertices of* s_j *in* C *are* $u^j_{i_j}$ *and* u^{j+1}_1*, for* $1 \le j \le t-1$*. The input is layout of separation at most* $k-1$ *for every child of every vertex in* C*, and a* (k)*-*$(u^j_{i_j}, u^{j+1}_1)$*-stretchable layout* M_j *for the* s_j*-constituent of* C*, for* $1 \le j \le t-1$*. The output is a layout* L *for* G *such that* $\mathsf{vs}_L(G) \le k$*.*

Construction. Using Method 2, build a (k)-$(u^j_1, u^j_{i_j})$-stretchable layout L_j for the p_j-constituent of C, for $1 \le j \le t$. In each L_j, the vertex u^j_1 is the leftmost vertex by construction. As we already said, Observation 1 allows us to consider without loss of generality that in each M_j the leftmost vertex is $u^j_{i_j}$. Define that for $1 \leq j \leq t-1$, $\overline{M_j} = M_j - u^j_{i_j}$, and for $2 \leq j \leq t$, $\overline{L_j} = L_j - u^j_1$. Consider the following layout L for G:

$$L = L_1, \overline{M_1}, \overline{L_2}, \overline{M_2}, \overline{L_3}, \dots, \overline{L_{t-1}}, \overline{M_{t-1}}, \overline{L_t}$$

The separation of the vertices from $\overline{L_t}$ under L is the same as it was under L_t. Each of the other layouts in L has a vertex, namely $u^1_{i_1}$ in L_1, $u^j_{i_j}$ in $\overline{L_j}$ for $2 \leq j \leq t-1$, and u^{j+1}_1 in $\overline{M_j}$ for $1 \leq j \leq t-1$, such that the respective layout is right-extended with respect to that vertex under L. However, each L_j is (k)--right($u^j_{i_j}$)-ext and each M_j is (k)-right(u^{j+1}_1)-ext by the premises of this method. Therefore, each $\overline{L_j}$ is (k)-right($u^j_{i_j}$)-ext and each $\overline{M_j}$ is (k)-right(u^{j+1}_1)-ext, so the separation of every vertex in L is at most k. □

Method 4. *The input is a noncritical rooted cactus G with root u such that $vs(G) = k$. The output is a k-compliant c-path in G with one endpoint u.*

Construction. Consider the following procedure on G, which uses a variable z of type vertex and a variable C of type c-path.

Initialise $C \leftarrow u$ and $z \leftarrow u$. While z has a child H of separation k, do:

1. If H is a tree child of z with root v, set $C \leftarrow C, v$ and $z \leftarrow v$.

2. Otherwise, let s be the parental cycle of H. Choose arbitrarily any vertex v from s, such that $v \neq z$ and $G[s] \ominus [z, v]$ is k-stretchable. Set $C \leftarrow C, s, v$ and $z \leftarrow v$.

As G is noncritical, vertex z can have at most one separation k child at any iteration, so cases 1 and 2 are exhaustive.

In case 2, a vertex v as specified there exists, because z is the root of s and by the premises the root is k-favourable in s. Furthermore, v has to be distinct from z because, if $z = v$ then $G[s] \ominus [z, z]$ is k-stretchable, so, by Observation 2, the arch of s relative to z is of separation $\leq k-1$; however, that arch is in fact H and $vs(H) = k$ by construction.

The iteration can be executed at most a number of times that is less than the number of vertices in G because the variable z is set to a new vertex at every iteration, *i.e.* a vertex that has not been considered before. So, the procedure halts, since G is finite.

We prove that the constructed C is k-compliant by induction on the number of times the loop is executed. Our loop invariant is the following claim: either the hitherto built C is k-compliant, or z, which is one endpoint of C, has a separation k child H—vertex-disjoint with C—and C is k-compliant in $G - H$. Clearly, this assertion holds before the iterations start, and it holds after each iteration given it holds before it. $\square$

4. Lemmas and Theorems.

4.1. Lemmas. The following lemma is from [6, Lemma 7, pp. 40]. However, there is a small "typo" error in its proof there: in the second sentence of the proof it says "$|\pi_{L_i}(u_i)| \leq k$". That must rather be "$|\pi_{L_i}(u_i)| \geq k$". So we do the proof here again.

Lemma 1 ([6])**.** *Let G be a connected graph of vertex separation $k > 1$. Let G_1, G_2, G_3 be connected, pairwise vertex-disjoint subgraphs of G, each one of them of vertex separation at least k, such that between any two G_i, G_j there is a path that is vertex-disjoint with the third one G_k. Then the vertex separation of G is at least $k+1$.*

P r o o f. Suppose that L is an optimal layout for G and L_i is the sublayout for G_i under L, for $i = 1, 2, 3$. By the premises, there is a vertex $u_i \in G_i$ such that $|\pi_{L_i}(u_i)| \geq k$. Without loss of generality, suppose that $L(u_1) < L(u_2) < L(u_3)$. By the premises, there is a path p between u_1 and u_3 that is vertex-disjoint with G_2. No vertex from p is in $\pi_{L_2}(u_2)$ but a vertex from p is in $\pi_L(u_2)$. Therefore, $\pi_{L_2}(u_2) \subset \pi_L(u_2)$. By the premises, $|\pi_{L_2}(u_2)| \geq k$, so $|\pi_L(u_2)| \geq k+1$. $\square$

Lemma 2. *Suppose that G is a connected graph and L is a layout for G such that $vs_L(G) \leq k$. Suppose that a is the leftmost vertex of L and z is the rightmost vertex. Suppose that H is a connected subgraph of G and u and v are two not necessarily distinct vertices in H. Suppose that $a \overset{p}{\frown} u$, $v \overset{q}{\frown} z$, $H \cap p = u$, and $H \cap q = v$. Further, $p \cap q = \emptyset$, in case that $u \neq v$, and $p \cap q = u$, in case that $u = v$. Under these assumptions, H is (k)-(u, v)-stretchable.*

P r o o f. We prove that L_H, the sublayout of H under L, is (k)-(u,v)--stretchable. First assume that u and v are distinct. Now assume that $L_H(u) < L_H(v)$. Let α and ω be the leftmost and the rightmost, respectively, vertex under L_H. Let L_H be broken into the following three intervals: $\mathcal{I}_1 = \alpha, L_H^{-1}(2), \ldots, L_H^{-1}(L_H(u) - 1)$, $\mathcal{I}_2 = u, L_H^{-1}(L_H(u) + 1), \ldots, L_H^{-1}(L_H(\text{right}(v)) - 1)$, and

$\mathcal{I}_3 = \text{right}(v), L_H^{-1}(L_H(\text{right}(v)) + 1), \ldots, \omega$, as illustrated here:

$$L_H = \underbrace{\alpha, \ldots \ldots}_{\mathcal{I}_1}, \underbrace{u, \ldots, v, \ldots \ldots}_{\mathcal{I}_2}, \underbrace{\text{right}(v) \ldots \ldots, \omega}_{\mathcal{I}_3}$$

Vertex v is shown to be distinct from right(v) but that does not have to be the case; if $v = \text{right}(v)$ then $\mathcal{I}_3 = v, \ldots, \omega$. Of course, by "right($v$)" we mean "right($v$) in L_H". Consider any vertex $y \in H$.

- If $y \in \mathcal{I}_1$, then $|\pi_{L_H}(y)| < k$ because for some vertex $x \in p$ such that $x \notin H$, $x \in \pi_L(y)$ and $x \notin \pi_{L_H}(y)$. Note that $\mathcal{I}_1$ is non-empty since y is in it, therefore u cannot be the leftmost vertex under L, therefore $u \neq a$ and so p has more than one vertex. All the vertices from p that are left of y under L are not from H and there is at least one such vertex, namely a.

- If $y \in \mathcal{I}_2$, then $|\pi_{L_H}(y)| \leq k$ by the premises of this lemma—it must be the case that $vs_{L_H}(H) \leq k$ as $vs_L(G) \leq k$.

- Suppose that $y \in \mathcal{I}_3$. If $\pi_L(y) = \emptyset$[2], certainly $|\pi_{L_H}(y)| < k$.

  Suppose that $\pi_L(y) \neq \emptyset$. Consider any vertex $x \in q$ that is in $\pi_L(y)$; such a one clearly exists. If $x = v$, *i.e.* $x \in H$, note that $v \notin \pi_{L_H}(y)$ by the definition of right(v). If $x \neq v$, then $x \notin \pi_{L_H}(y)$ because $x \notin H$. In any event, $x \notin \pi_{L_H}(y)$ and $x \in \pi_L(w)$, therefore $|\pi_{L_H}(w)| < k$. This result remains valid even in case that $v = \text{right}(v)$, because then v does not contribute to its own separation under L_H but contributes to its own separation under L.

So, the separation of any vertex from $\mathcal{I}_1$ and $\mathcal{I}_3$ under L_H is at most $k - 1$, and the separation of any vertex from $\mathcal{I}_2$ under L_H is at most k. According to Definition 5, L_H is (k)-(u, v)-stretchable.

Now assume that $L_H(v) < L_H(u)$. Suppose that L_H is broken into three intervals $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ in the same way as above. However, now v is left of u. If u is left of right(v):

$$L_H = \underbrace{\alpha, \ldots, v, \ldots}_{\mathcal{I}_1}, \underbrace{u, \ldots \ldots}_{\mathcal{I}_2}, \underbrace{\text{right}(v), \ldots, \omega}_{\mathcal{I}_3}$$

[2]It is possible $\pi_L(y)$ to be empty. The only way this can happen is y to be the rightmost vertex of L, *i.e.* $y = z$. But y is in H and the only common vertex between the path q and H is v, so it has to be the case that q consists of a single vertex. Then v is the rightmost vertex of L, and v coincides with right(v).

we deduce that L_H is (k)-(u,v)-stretchable in exactly the same way as above, because v being between a and u changes nothing in this case.

If $u = \text{right}(v)$, the interval $\mathcal{I}_2$ is empty (since $\mathcal{I}_3 = u, \ldots, z$), but our proof remains valid because the vertices from $\mathcal{I}_1$ and $\mathcal{I}_3$ have separation at most $k-1$, just as before.

Now suppose that $L_H(v) < L_H(u)$ and right(v) is left of u. Let L_H be broken into the following three intervals: $\mathcal{I}_1 = \alpha, L_H^{-1}(2), \ldots, L_H^{-1}(L_H(\text{right}(v)) - 1)$, $\mathcal{I}_2 = \text{right}(v), L_H^{-1}(L_H(\text{right}(v)) + 1), \ldots, L_H^{-1}(L_H(u) - 1)$, and $\mathcal{I}_3 = u, L_H^{-1}(L_H(u) + 1)$, $\ldots, \omega$, as illustrated here:

$$L_H = \underbrace{\alpha, \ldots, v, \ldots}_{\mathcal{I}_1}, \underbrace{\text{right}(v), \ldots\ldots}_{\mathcal{I}_2}, \underbrace{u, \ldots\ldots, \omega}_{\mathcal{I}_3}$$

Again, vertex v is shown to be distinct from right(v) but that does not have to be the case; if $v = \text{right}(v)$ then $\mathcal{I}_1$ has its rightmost vertex just to the left of v. And again, by "right(v)" we mean "right(v) in L_H". Consider any vertex $y \in H$.

- If $y \in \mathcal{I}_1$ or $y \in \mathcal{I}_2$, then $|\pi_{L_H}(y)| < k$ because for some vertex $x \in p$ such that $x \notin H$, $x \in \pi_L(y)$ and $x \notin \pi_{L_H}(y)$.
- If $y \in \mathcal{I}_2$ or $y \in \mathcal{I}_3$, then $|\pi_{L_H}(y)| < k$ because for some vertex $x \in q$, $x \in \pi_L(y)$ and $x \notin \pi_{L_H}(y)$. That remains true even if $x = v$ because v does not contribute to the vertices from $\mathcal{I}_2$ and $\mathcal{I}_3$ in L_H.
- If $y \in \mathcal{I}_2$, the above two considerations apply independently. That is, there is a vertex from p contributing to the separation of y under L but not under L_H and there is a vertex from q contributing to the separation of y under L but not under L_H, and those two vertices are different. Therefore, $|\pi_{L_H}(y)| \leq k - 2$.

So, the separation of any vertex from $\mathcal{I}_1$ and $\mathcal{I}_3$ under L_H is at most $k-1$, and the separation of any vertex from $\mathcal{I}_2$ under L_H is at most $k-2$. According to Definition 5, L_H is (k)-(u,v)-stretchable.

It remains to consider the case that $u = v$. It is easy to see that the separation of any vertex from $H - u$ under L_H is at most $k-1$, regardless of whether u is distinct from a and z, or coincides with one of them. $\square$

Lemma 3. *Suppose that* G *is a cactus,* s *is a cycle in it, and* u *and* v *are not necessarily distinct vertices in* s*. Suppose that* H *is any arch relative to* u, v *and* $G[s] \ominus [u,v]$ *is* (k)-(u,v)-*stretchable. Then* $vs(H) < k$.

P r o o f. First assume that $u \neq v$. Assume that L is k-stretchable with respect to u and v layout for G and, without loss of generality, assume that $L(u) = 1$. Then L is (k)-right(v)-ext. Note that $u \overset{p}{\smile} v$, such that p is vertex-disjoint with H. That is true even if H is the only arch relative to u, v, *i.e.* if u and v are adjacent. Call L_H the sublayout of H under L and assume that $vs_{L_H}(H) \geq k$. Let z be any vertex from H that has separation at least k under L_H. Note that z cannot possibly be to the right of, or coincide with, right(v), by Definition 2. Then note that z cannot be between v and right(v), because v contributes to the separation of all vertices in that interval but $v \notin H$ and therefore $v \notin \pi_{L_H}(z)$; if z were between v and right(v) then $|\pi_L(z)|$ would be more than k.

So, z has to be between u and v. But for at least one vertex x from p, $x \in \pi_L(z)$ and $x \notin \pi_{L_H}(z)$, which implies that $|\pi_L(z)| > k$. It follows that our assumption that $vs_{L_H}(H) \geq k$ must be wrong.

Now assume that $u = v$. Then $G[s] \ominus [u, v]$ is in fact $G[s] \ominus [u, u]$ and so $G[s] \ominus [u, v]$ is (k)-(u, u)-stretchable. By Observation 2 on page 80, $vs(G[s] \ominus [u, u] - u) \leq k - 1$. Since H is a subgraph of $G[s] \ominus [u, u] - u$, it must be the case that $vs(H) \leq k - 1$. □

4.2. The Main Theorem for vertex separation on cacti. The following theorem generalises Theorem 3.1 from [3], the main theorem for the vertex separation of trees.

Theorem 1. *Suppose that* G *is a cactus and* $k \geq 1$. *Then* $vs(G) \leq k$ *if and only if both of the following hold:*

1. *for every vertex* $u \in G$*, at most two subcacti begotten by* u *have separation* k*, and all the other begotten subcacti have separation at most* $k - 1$.
2. *every cycle in* G *has a* k*-favourable pair.*

P r o o f o f n e c e s s i t y. Assume that $vs(G) \leq k$ and L is a layout for G such that $vs_L(G) \leq k$. Consider any vertex $u \in G$. It is obvious that no subcactus begotten by u can be of separation $> k$. By Lemma 1, there cannot be more than two subcacti begotten by u of separation k. So, condition 1 holds.

Now we prove that condition 2 holds. Consider any cycle s in G. Let v_1, v_2, …, v_t be the vertices of s. Let a be the leftmost vertex under L and z be the rightmost one. Say, a is in the v_i-constituent and z is in the v_j-constituent of s. Define that $H = G \ominus [v_i, v_j]$. We prove that v_i and v_j are a k-favourable pair. It may or may not be the case that a coincides with v_i, and likewise for z and v_j. Our proof holds anyways.

First assume that $v_i \neq v_j$. Apply Lemma 2 with the current H, v_i, and v_j as H, u, and v, respectively, from the lemma. Conclude that H is (k)-(v_i, v_j)--stretchable.

Now assume that $v_i = v_j$. Suppose that $a \overset{p}{\smile} v_i$ and $v_i \overset{q}{\smile} z$, such that $p \cap q = \{v_i\}$. Again, Lemma 2 implies that H is (k)-(v_i, v_i)-stretchable.

It remains to consider the case when $v_i = v_j$ and for each two paths p and q such that $a \overset{p}{\smile} v_i$ and $v_i \overset{q}{\smile} z$, $|p \cap q| \geq 2$. But then there exists a path p' between a and z, such that $p' \cap H = \emptyset$. It follows that p' contributes to the separation of any vertex from H under L, therefore $vs_{L_H}(H) < k$, and so H is trivially (k)-(v_i, v_i)-stretchable.

P r o o f o f s u f f i c i e n c y. Suppose that premises 1 and 2 hold. The crux of the proof is to construct a k-compliant c-path C in G. Using C, Method 3 constructs a separation k layout L for G.

Case I: There are important vertices or cycles in G. We choose arbitrarily one such vertex or cycle as our "starting point".

Case I.a: We choose some k-important vertex u and make G a rooted cactus with root u. Then u has precisely two children, call them H_1 and H_2, of separation k, and all the other children are of separation less than k. Define that $G_1 = H_1 + u$ and $G_2 = H_2 + u$. Note by Definition 9, both G_1 and G_2 are rooted cacti with root u. We claim that both G_1 and G_2 are noncritical. Assume the opposite: say, G_1 is critical.

- Assume that some vertex $v \in G_1$ has two children G' and G'' of separation k. Think of G as a non-rooted cactus and note that v begets three separation k subcacti in it: namely, G', G'', and the subcactus containing H_2. That contradicts premise 1 of this theorem.

- Assume that for some cycle s in G_1, its root v is not k-favourable, where v may possibly coincide with u. Think of G as a non-rooted cactus and note that by premise 2 applied to G, there are vertices x, y in s such that $G[s] \ominus [x, y]$ is (k)-(x, y)-stretchable; however, $v \notin \{x, y\}$ because v is not k-favourable in s. But then, in the context of G, v is in an arch of s relative to x and y. Call that arch J and note that $vs(J) \geq k$ because J contains H_2 as a subcactus and $vs(H_2) = k$. Apply the contrapositive of Lemma 3 and conclude that $G[s] \ominus [x, y]$ is not (k)-(x, y)-stretchable. The result holds even when $x = y$.

So, none of G_1, G_2 is critical. Method 4 constructs a k-compliant c-path C_1 in G_1 such that u is one endpoint of C_1, and a k-compliant c-path C_2 in G_2 such that u is one endpoint of C_2. Define that $\overline{C_2} = C_2 - u$ and observe that $C_1, \overline{C_2}$ is a k-compliant c-path in G.

Case I.b: We choose some important cycle s. There must exist a favourable pair of vertices u_1, u_2 in s by premise 2. Define that G_i is the u_i-constituent of s and make G_i a rooted cactus with root u_i, for $i = 1, 2$. Also, define that H_i is the arch of s relative to u_i, for $i = 1, 2$. By the definition of "k-important cycle", $vs(H_1), vs(H_2) = k$. We claim that for $i = 1, 2$, either $vs(G_i) < k$ or $vs(G_i) = k$ and G_i is noncritical. Assume the opposite: say, G_1 has separation k and is critical.

- Assume that some vertex $v \in G_1$ has two children G' and G'' of separation k. But then v begets three separation k subcacti in G: namely, G', G'', and the subcactus containing H_2. That contradicts premise 1 of this theorem.

- Assume that for some cycle s' in G_1, its root v is not k-favourable, where v may possibly coincide with u_1. Then by premise 2 applied to G, there are vertices x, y in s' such that $G[s'] \ominus [x, y]$ is (k)-(x, y)-stretchable; however, $v \notin \{x, y\}$ because v is not k-favourable in s. But then, in the context of G, v is in an arch of s' relative to x and y. Call that arch J and note that $vs(J) \geq k$ because J contains H_2 as a subcactus and $vs(H_2) = k$. Apply the contrapositive of Lemma 3 and conclude that $G[s] \ominus [x, y]$ is not (k)-(x, y)-stretchable. The result holds even when $x = y$.

So, none of G_1, G_2 is of separation k and critical. For any G_i, $i = 1, 2$, if $vs(G_i) = k$ then Method 4 constructs a k-compliant c-path C_i in it such that u_i is one endpoint of C_i; and in $vs(G_i) < k$ then u_i is a k-compliant c-path C_i in it. Define that $\overline{C_2} = C_2 - u_2$. Further, notice that the c-path $C_3 = u_1, s, u_2$ is k-compliant in $G[s] \ominus [u_1, u_2]$ by the assumptions about s. Define that $\overline{C_3} = C_3 - u_1$, *i.e.* $\overline{C_3} = s, u_2$. Observe that $C_1, \overline{C_3}, \overline{C_2}$ is a k-compliant c-path in G.

Case II: There are no important vertices and there are no important cycles in G. We prove that there is a k-compliant c-path that consists either of a single vertex or of two vertices.

Let a be any vertex in G. Consider the following procedure on G. It uses variables x and y of type vertex and a variable C of type c-path.

```
Initialise x ← a, y ← x and iterate:

  1. If all subcacti begotten by x are of separation less
     than k, set C ← x and stop.
```

2. `Otherwise, let` H `be the separation` k `subcactus begotten by` x`.`
   (a) `If` x `is connected to` H `by a tree edge and that edge is` (x, y)`, set` $C \leftarrow x, y$ `and stop.`
   (b) `If` x `is connected to` H `by a tree edge` (x, z) `where` $z \neq y$`, set` $y \leftarrow x$`,` $x \leftarrow z$ `and continue iterating.`
   (c) `Otherwise, let` s `be the cycle that contains` x `and vertices from` H`. Let` w `be a vertex from` s `such that the arch of` s `relative to` w `is of separation less than` k`. Set` $y \leftarrow x$`,` $x \leftarrow w$ `and continue iterating.`

`return` C

Observe that cases 1, 2a, 2b, and 2c exhaust all possibilities: by the current premises, any vertex begets at most one separation k subcactus, so the division into cases 1 and 2 is exhaustive, and if x begets a separation k subcactus H, that H can be connected to x by a tree edge (cases 2a and 2b) or by two cycle edges (case 2c).

Now we show the procedure halts. Let us denote by $C_{b,c}$ the (unique) c-path connecting any two vertices b and c in G. Observe that y is a vertex from $C_{a,x}$; furthermore, x and y are c-path neighbours in $C_{a,x}$ after the first iteration because then y has the former value of x. It clearly follows that $C_{a,x}$ gets incremented at its "x end" every time the procedure enters case 2b or case 2c. Consider that G is finite and conclude that the procedure halts.

If the procedure halts in case 1 it is obvious that the returned c-path consists of the single vertex x and is k-compliant. Suppose that the procedure halts in case 2a, *i.e.* after an attempt to "go back". Then x and y are connected by a tree edge, all the subcacti begotten by y have separation $< k$ except for the subcatus containing x (otherwise, the procedure would have already stopped), and all the subcacti begotten by x have separation $< k$ except for the subcactus containing y (otherwise, the procedure would be in case 1). So, the c-path x, y is k-compliant since the children of every vertex in it are of separation at most $k - 1$. □

5. On the stretchability of cacti. In this section we consider a cactus G as illustrated on Figure 2: there is a cycle s in it such that two vertices

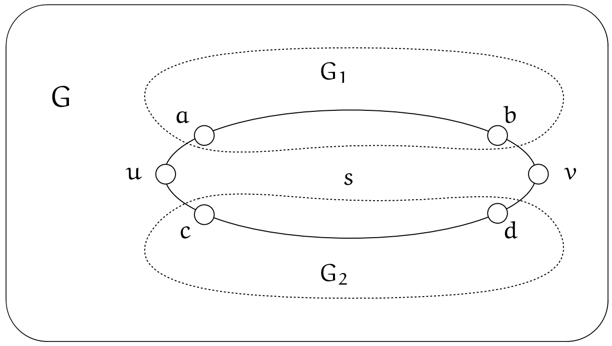


Fig. 2. The cactus G in Section 5

u and v from s are of degree two. We want to answer the question whether G is k-stretchable with respect to u and v.

We assume the reader is familiar with [2][3] and especially Section 4: it answers precisely the same question for unicyclic graphs (compare Figure 2 with [2, Figure 9]). Our approach is to follow a line of reasoning as close as possible to the one in [2, Section 4].

The archs of s relative to u and v are G_1 and G_2. We call them simply "the archs". Consider the general case when both archs are non-empty. Whether or not G is (k)-(u, v)-stretchable clearly depends on, and only on, the properties of the archs. Let us call the vertices u and v, *the special vertices*, and the other vertices from G, *the ordinary vertices*. Consider any layout L for G in general. Precisely one of the following two possibilities is the case for L.

p1: The leftmost and the rightmost ordinary vertex are from the same arch.

p2: The leftmost and the rightmost ordinary vertex are from different archs.

A feasible approach to compute whether G is k-stretchable is the following: identify necessary and sufficient conditions for the existence of k-stretchable layout for G for **p1** and for **p2**, and then test G under both conditions. G is k-stretchable whenever at least one of those tests gives a positive answer.

Note that [2, Section 4] takes, though implicitly, the same approach. It considers three cases among which only Case 3 is relevant because Case 1 simply gives a negative answer when one of the two archs of s has separation $> k-1$, and Case 2 gives a trivial positive answer when the separation of both archs is $\leq k-2$. The said Case 3 subdidivides into Case 3.1 and Case 3.2. Those two subcases

[3]A copy is available at http://www.cs.uvic.ca/~jellis/Publications/unicyclic.ps

are specified by the separations and the criticalities or the noncriticalities of the archs T_1 and T_2 but in fact Case 3.1 explores the possibility that the leftmost and the rightmost ordinary vertex are from the same arch (namely T_1) and Case 3.2 is about the alternative possibility. To see why that is true, consider the following:

- in Case 3.1, if there is a layout then it is in one of the three possible forms shown on Figures 12, 13, and 16: both the rightmost and the leftmost ordinary vertices are (*i.e.,* not r_i or r_j in the naming convention of [2, Section 4]) from T_1;
- in Case 3.2, if there is a layout then it is as shown on Figure 17: the rightmost ordinary vertex is from T_2 and the leftmost, from T_1.

Let us try to generalise the results from [2, Section 4, Case 3] for the cactus G that we currently consider.

5.1. Generalising case 3.2 for cacti. Recall that [2] uses the notation "$T[z]$", where T is a tree and z is a vertex in it, to mean T rooted at z. We can extend that notation to cacti in the obvious way.

The said Case 3.2 in [2] considers a unicyclic graph such that $vs(T_1) = k-1$, at least one of $T_1[a]$ and $T_1[b]$ is not $(k-1)$-critical, and at least one of $T_2[c]$ and $T_2[d]$ is not $(k-1)$-critical. The vertices a, b, c, and d in Case 3.2 are defined by Figure 9 of [2]. Our current Figure 2 defines them for the current G likewise.

Case 3.2 defines[4] that T_1 and T_2 have *complementary extensibilities* if $T_1[a]$ and $T_2[d]$ are not $(k-1)$-critical or $T_1[b]$ and $T_2[c]$ are not $(k-1)$-critical, and proves that:

- the unicyclic graph U has a k-stretchable layout such that the extreme ordinary vertices are from different archs if T_1 and T_2 have complementary extensibilities (see Figure 17);
- there is no k-stretchable layout for G if their extensibilities are not complementary.

So, effectively, Case 3.2 proves that there is a k-stretchable layout such that the extreme ordinary vertices are from different archs if and only if the extensibilities of the archs are complementary.

[4] Note that the definition of complementary extensibilities in Case 3.2 of [2][Section 4] has a slight imprecision: it uses the language construct "... either ... or ..." that suggests exclusive "or", while in fact it should be ordinary, inclusive "or".

Now compare criticality of trees as defined in [3] and used in [2], on the one hand, and criticality of cacti as defined in the current paper, on the other hand. It is obvious that the latter is a generalisation of the former because our Definition 17 reduces to Definition 3.1 from [3] when the cactus in consideration is a tree. However, there is a deeper connection between them, namely that:

- a separation k rooted tree is k-extensible with respect to its root if and only if it is not k-critical;
- a separation k rooted cactus is k-extensible with respect to its root if and only if it is not k-critical.

The first fact is easily deducable from [2] and is proved explicitly in [6] (see Lemma 2, pp. 35, Lemma 4, pp. 36, and Corollary 3, pp. 40). The second fact is not difficult to prove. In one direction, it follows from Method 4. We do not provide a proof for the other direction because that would be distracting to the goals of this section; the reader is invited to try to make one.

Having in mind the deeper connection between criticality of trees and criticality of cacti, we can define complementary extensibilities for our current archs G_1 and G_2 just as in Case 3.2, namely that G_1 and G_2 have complementary extensibilities whenever $G_1[a]$ and $G_2[d]$ are not $(k-1)$-critical or $G_1[b]$ and $G_2[c]$ are not $(k-1)$-critical, and prove that G has a k-stretchable layout such that the leftmost and rightmost ordinary vertex are from different archs if and only if the archs have complementary extensibilities. The proof can be done just as in Case 3.2 from [2] precisely because criticalities are equivalent to non-extensibilities for both trees and cacti.

5.2. Trying to generalise Case 3.1: a failure. Let us consider how Case 3.1 from [2] relates to cacti and in particular the subcase when there exists one constituent tree of the cycle with a buried $(k-1)$-critical vertex (the other two subcases are easy to generalise for cacti; the reader is invited to check that, having in mind what we discussed about how to generalise Case 3.2). Let us define that the critical vertex in a rooted cactus is buried if it is not the root. It is not hard to see that if there is a $(k-1)$-critical vertex in, say, G_1, we can "excise" the subcactus rooted at it (within the corresponding constituent of s) obtaining some cactus G' and then there exists a (k)-(u,v)-stretchable layout for G if and only if there exists a $(k-1)$-(u,v)-stretchable layout for G'. In other words, the results from [2, Section 4] are once again directly generalisable for cacti.

However, the said subcase of Case 3.1 suggests a possibility for cacti that has no analogy for unicyclic graphs. Suppose that G_1 is of separation $k-1$ and contains a $(k-1)$-critical constituent that has a $(k-1)$-critical cycle. That can happen if, for instance, G is as suggested by Figure 3: for some vertex w from s, there is a cycle s_1 in the w-constituent of s, such that—assuming that constituent is rooted at w—the root of s_1 is some vertex f and there are two vertices x and y in s_1, $x, y \neq f$, such that there is a separation $k-1$ subcactus "hanging off" both x and y. It is clear that if $vs(H_1), vs(H_2) = k-1$ then f is indeed a $(k-1)$-unfavourable vertex, thus s_1 is critical. For simplicity, assume that H_1 is the only child of x in s_1, and the same for H_2 and y.

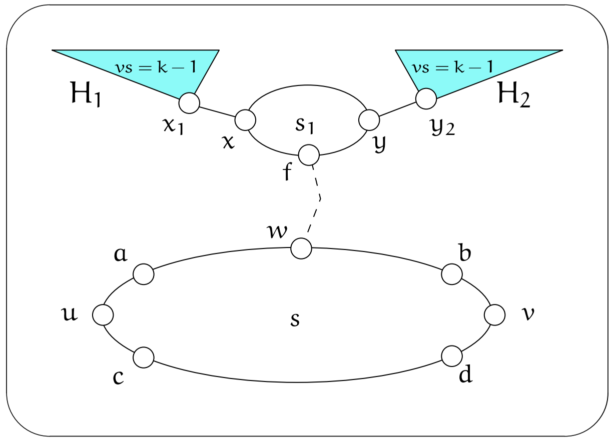


Fig. 3. A possibility with no analogy for unicyclics: the two "buried" separation $k-1$ subgraphs are attached to a cycle, not a vertex, within one arch of s. Compare with [2, Figure 11]

Before we proceed with the signicance of the example on Figure 3, we explain semi-formally what is stretchability with respect to *two* pairs of vertices. It is known that the problem Vertex Separation is equivalent to Node Search Number in the sense that $vs(G) = k$ if and only if $nsn(G) = k+1$ for any graph G (see [5]). The relation is in fact deeper since a linear layout determines a node search strategy and *vice versa*. The stretchability property, for instance, can be expressed in terms on node searching: G is k-stretchable with respect to some vertices u and v if and only if there is a search with $k+1$ searchers that starts on u and finishes on v (see [6]). We express our notion of *stretchability with respect to two vertex pairs* in terms of node searching. Let us use the notation "$\{u, v\}$" to designate the multiset[5] of the vertices u and v. If u, v, x, and y are any four

[5]Note that it is not necessary the vertices to be distinct.

vertices in some graph G then G is k-*stretchable with respect to* $\{u, v\}$ *and* $\{x, y\}$ if there is a search using $k+1$ searchers that either starts on u and x and finishes on v and y, or starts at u and y and finishes at v and x. Analogously to the stretchability with respect to a single pair of vertices, it does not matter which vertices are associated with the left and which, with the right direction.

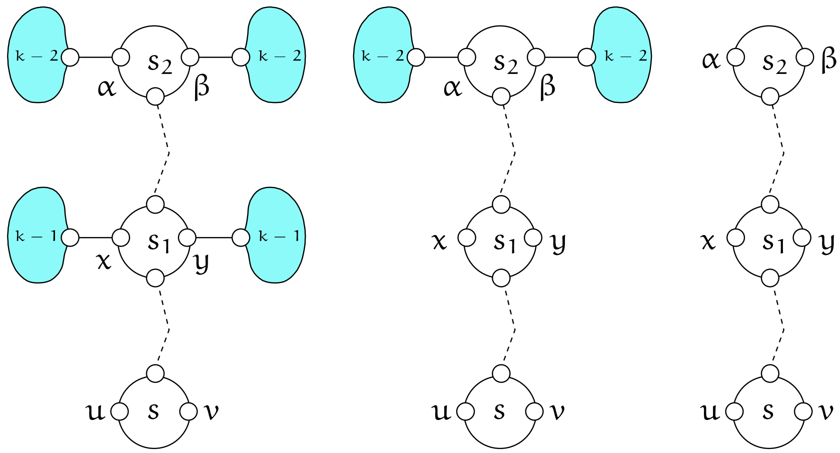
Going back to our cactus G as shown on Figure 3, it is not difficult to prove that G is (k)-(u, v)-stretchable if and only if $J = G[s_1] \ominus [x, y]$ is k-stretchable with respect to $\{u, v\}$ and $\{x, y\}$. The proof in one direction is: suppose that a k-stretchable with respect to $\{u, v\}$ and $\{x, y\}$ layout L_J exists and in it, u and x are associated with the left direction and v and y, with the right one. Then the layout $L = L_1, L_J, L_2$ for G, where L_1 is a (k)-right(x_1)-ext layout for H_1 and L_2 is a (k)-left(y_2)-ext layout for H_2, is indeed (k)-(u, v)-stretchable. Such L_1 and L_2 exist, but we are not concerned with proving that now, and we do not provide a proof for the other direction.

Compare the expression $L = L_1, L_J, L_2$ with the layout from [2, Figure 12]: they are completely analogous! The crucial difference is that in the unicyclic case, the sublayout of U' is stretchable with respect to the original pair of special vertices $\{r_i, r_j\}$, while in the current case, another pair of special vertices shows up, namely $\{x, y\}$.

By itself, the addition of another vertex pair necessitates theoretical results concerning stretchability with respect to two pairs of vertices. That sounds like a daunting task, having in mind the effort it took in [2, Chapter 4] to pinpoint the necessary and sufficient conditions for stretchability with respect to a single pair. The worst thing is that the stretchability with respect to two pairs may reduce to stretchability with respect to *three* vertex pairs.

We define stretchability with respect to three vertex pairs as a generalisation of the current definitions for stretchability. Suppose that u_1, v_1, u_2, v_2, u_3, v_3 are not necessarily distinct vertices in some graph G with the restriction that every two of the three pairs (u_i, v_i) have no vertex in common. G is k-*stretchable with respect to* (u_1, v_1), (u_2, v_2), *and* (u_3, v_3) if there exist two (not necessarily disjoint) vertex sets S_1 and S_2, each set having precisely one vertex from each pair, so that there exists a search on G with $k+1$ searchers starting at the vertices from S_1 and finishing at the vertices from S_2. Obviously, we can extend that definition to arbitrarily many vertex pairs.

Figure 4 gives a general idea about why we may have to add a third pair of special vertices. Suppose that G from Figure 3 is as shown on Figure 4.1: the arch of s_1 relative to x and y that does not contain s has a cycle s_2 such that for two vertices α and β from s_2, there are separation $k-2$ subcacti “hanging off” them. Figure 4.2 illustrates $J = G[s_1] \ominus [x, y]$. Intuitively, it is clear that J is k-stretchable with respect to (u, v) and (x, y) if and only if $K = J[s_2]\ominus[\alpha, \beta]$ (see Figure 4.3) is k-stretchable with respect to (u, v), (x, y), and (α, β).



4.1: The original G. 4.2: $J = G[s_1]\ominus[x, y]$. 4.3: K.

Fig. 4. G is k-stretchable with respect to (u, v) if and only if $J = G[s_1]\ominus[x, y]$ is k-stretchable with respect to (u, v) and (x, y). In its turn, that reduces to whether $K = J[s_2]\ominus[\alpha, \beta]$ is k-stretchable with respect to (u, v), (x, y), and (α, β)

It is obvious that process can go on and on: the k-stretchability of a cactus with respect to m vertex pairs may depend on the k-stretchability with respect to $m + 1$ pairs of vertices of a smaller cactus.

6. Conclusions. We have identified and verified a "main theorem for cacti": a necessary and sufficient condition for the separation of a cactus being k, that condition being expressed in terms of separations and stretchabilities of subcacti. The condition is a generalisation of the "main theorem for trees" [3, Theorem 3.1]. Unlike the main theorem for trees, however, it does not lead immediately to a fast, practical algorithm for the VS of cacti because the stretchability with respect to a single vertex pair reduces in general to stretchability with respect to two vertex pairs, which in its turn reduces in general to stretchability with respect to three vertex pairs, and so on.

REFERENCES

[1] Bodlaender H. L., T. Kloks. Efficient and Constructive Algorithms for the Pathwidth and Treewidth of Graphs. *J. Algorithms* **21** (1996), No. 2, 358–402.

[2] Ellis J., M. Markov. Computing the Vertex Separation of Unicyclic Graphs. *Information and Computation* **192** (2004), 123–161.

[3] Ellis J. A., I. H. Sudborough, J. S. Turner. The Vertex Separation and Search Number of a Graph. *Information and Computation* **113**, No 1 (1994), 50–79.

[4] Gibbons A. Algorithmic Graph Theory. Cambridge University Press, 1985.

[5] Kirousis L. M., C. H. Papadimitriou. Searching and pebbling. *Theoretical Computer Science* **47**, No. 2 (1986), 205–218.

[6] Markov M. A Fast Practical Algorithm for the Vertex Separation of Uncyclic Graps. Master's thesis, University of Victoria, Dec. 2004.

[7] Skodinis K. Computing optimal strategies for trees in linear time. Proceedings of the 8th Annual European Symposium on Algorithms, 2000, 403–414.

Department of Mathematical Linguistics
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: minko@cc.bas.bg

Received October 5, 2006
Final Accepted February 23, 2007