Serdica J. Computing **1** (2007), 425–432

Serdica
Journal of Computing

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

SYMBOLIC DYNAMICS IN THE FREE-FALL EQUAL-MASS THREE-BODY PROBLEM*

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Abstract. Free-fall equal-mass three-body systems are numerically studied using symbolic dynamics. We scan the two-dimensional homology map of initial configurations in steps of 0.001 along both axes. States of binary and triple encounters as well as changes of configuration are used to construct symbolic sequences. Symbolic sequences are characterized by Shannon and Markov entropies. Different ergodic components corresponding to different distinct peaks on the histograms of these entropies are revealed.

1. Introduction. The gravitational three-body problem is a well-known problem of mathematics and celestial mechanics. It has a very simple formulation (see, e.g., [10]). However it is very difficult for analytical solution. In the literature, there are many various approaches to its study (see, e.g., [18]). One of the approaches is using symbolic dynamics.

ACM Computing Classification System (1998): J.2.
Key words: Three-body problem, symbolic dynamics.

*The paper has been presented at the 12th International Conference on Applications of Computer Algebra, Varna, Bulgaria, June, 2006.

The symbolic dynamics approach is based on the topological conjugacy between the continuous evolution of the dynamical system and a shift map on the space of sequences of integer numbers reflecting the state of the evolution. Constructed symbolic sequences allow one to describe various orbit types (periodic orbits, their bifurcations, triple collisions etc.).

Alexeyev [6] has applied symbolic dynamics to one special case of the three-body problem (Sitnikov's problem [15]). Using the symbolic dynamics approach, Alexeyev [3, 4, 5, 6] found an intermittence of motions of different types in the Sitnikov problem.

Symbolic dynamics was also applied in two other special cases of the three-body problem: the rectilinear problem ([16, 17],[13, 14]); and the isosceles problem ([19]; [9]).

We have also started a similar study for the free-fall equal-mass three-body problem ([12], [8]). The present work continues these studies.

2. Methods and Results. A symbolic dynamical system consists of three basic parts: an alphabet Ω, a space X of infinite sequences

$$\{\omega_i\},\ i \in Z,\ \omega_i \in \Omega,$$

and a shift transformation σ that moves (shifts) the sequence one position to

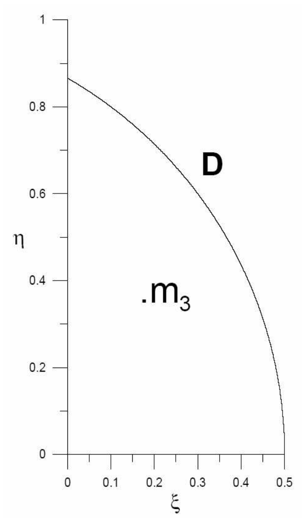

Fig. 1. The homology region D

the left. We construct symbolic sequences for the free-fall equal-mass three-body problem in several different ways. One approach consists of introducing some partition of the phase space and fixing in what subregion the trajectory is at the current moment. Since not all transitions between subregions are possible, this approach corresponds to subshifts of finite type. We use a homology region D ([2], see Fig. 1) for this partitioning. It was used to specify initial conditions so as to consider all possible configurations: two bodies are placed in the points $(-0.5, 0)$ and $(0.5, 0)$, then to consider all possible geometric configurations, the third body should be placed inside the region D (Fig. 1). This region is bounded by the coordinate axes and the circle of unit radius with center in the point $(-0.5, 0)$. For each triangle there is a similar triangle when the third component is placed inside the region D. The system is projected to the region D according to the relative distances between bodies. There are six different projections possible, thus we get sequences constructed from the alphabet $\{1, 2, 3, 4, 5, 6\}$. A second approach is to fix some dynamical states (double encounters, triple encounters etc.) during the evolution of the triple system. We used double and triple encounters to construct two more sequences with alphabets $\{1, 2, 3\}$.

We characterized symbolic sequences by Shannon (H_1) and Markov (H_2) entropies:

$$H_1 = -\sum_i p_i \ln p_i, \;\; H_2 = -\sum_i p_i \sum_j q_{ij} \ln q_{ij}.$$

Here p_i is the frequency of symbol "i" in the sequence, and q_{ij} is the frequency of transitions from "i" to "j". When we consider the double and triple encounters, we take ω_i as the number of the distant body at the moment of the

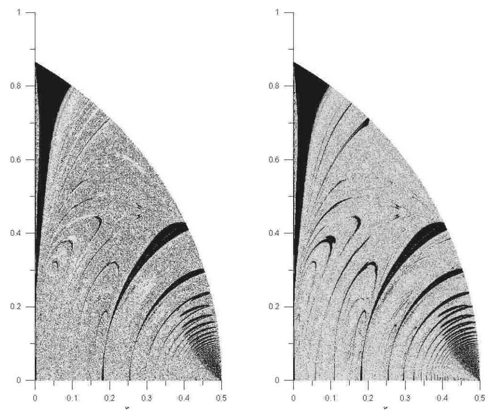

Fig. 2. Entropies H_1 (left) and H_2 (right) for triple encounters

closest double or triple encounter.

We have estimated the entropies H_1 and H_2 along the whole trajectory and fixed their maximum values. The results for triple encounters are shown in Fig. 2. Different shades of grey color correspond to different ranges of entropies' values (darker regions correspond to smaller values of the entropy).

On the histograms for the entropies and for maximum values of entropies, one can see that the distribution is far from uniform. There are strong peaks (Figures 3–5). Figure 6 shows initial conditions that correspond to two distinct peaks on the histogram for $H_{1\,\max}$ for triple encounters.

Our numerical experiments were made using the code TRIPLE by Sverre Aarseth (University of Cambridge, UK). This code uses regularization by Aarseth and Zare [1]. We scanned the two-dimensional homology region D of initial configurations in steps of 0.001 along both axes. Following post-processing of the results of the numerical simulations was done with extensive use of Mathematica.

3. Discussion. Points from different peaks form complicated structures in homology maps (see Fig. 6). Some of them are attracted to the point (0.4, 0.4). This property may reflect a role of "figure-of-eight" periodic orbit in the free-fall three-body problem. The significant part of runs (up to 50% of the initial positions) correspond to the peaks revealed. Thus, ergodic components that correspond to these peaks occupy a large fraction of the homology region D. The rest of the initial data gives a smoothed distribution on the histograms of entropies. This differentiation may reflect some features of the three-body dynamics. Future investigations would shed light on the reasons for this differentiation.

It is of interest to compare the histograms of entropies H_1 and H_2 versus histograms for their maxima (Figures 3–5). Some of the most prominent peaks coincide, whereas other peaks can be shifted or disappear altogether in one of histograms. This may be connected with variety of the behavior during the evolution of the system. Possibly, the peaks for smaller entropies correspond to shorter escape times, whereas the peaks for larger entropies correspond to the longer ones. This hypothesis demands a more detailed study. The number of prominent ergodic components is not so big. Usually, a few (sometimes two) higher peaks can be seen, and about ten lower peaks (a forest of peaks) could be revealed in smoothed entropy distribution. The statistical significance of this forest may be one of subjects of further studies. So symbolic dynamics can help us to study complicated dynamical systems such as the three-body problem.

Though we can't strictly prove the existence of the topological conjugacy and thus the validity of the symbolic dynamics approach for the general three-body problem, there are some experimental results in favor of using symbolic

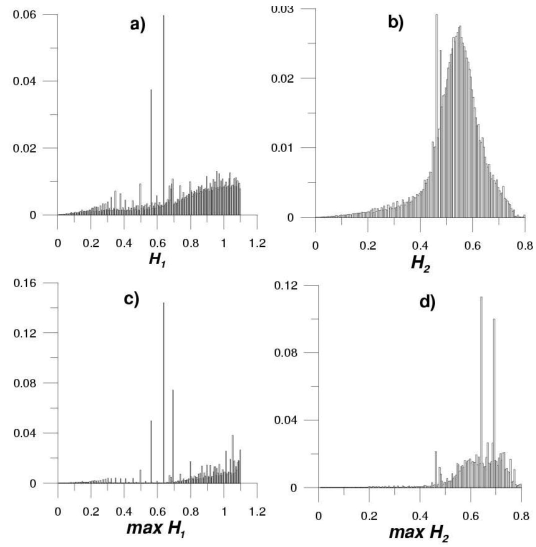

Fig. 3. Up: histograms for H_1 (left) and H_2 (right) for the partitioning using region D. Below: histograms for maximum values of H_1 and H_2

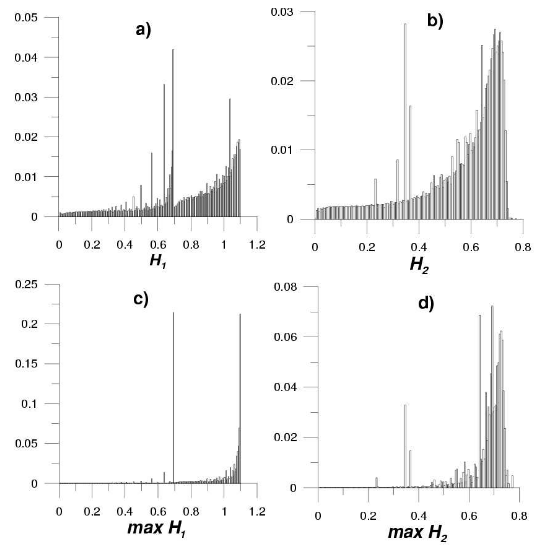

Fig. 4. Histograms for the entropies $H_1(\xi, \eta)$ and $H_2(\xi, \eta)$ for double encounters

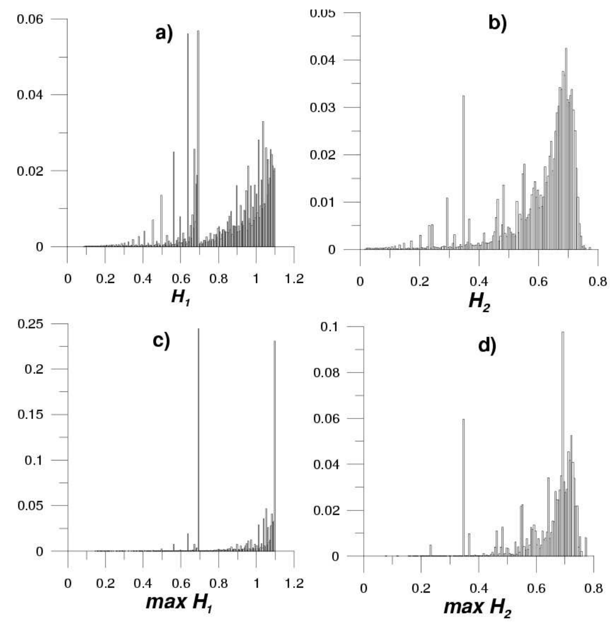

Fig. 5. Histograms for the entropies $H_1(\xi,\eta)$ and $H_2(\xi,\eta)$ for triple encounters

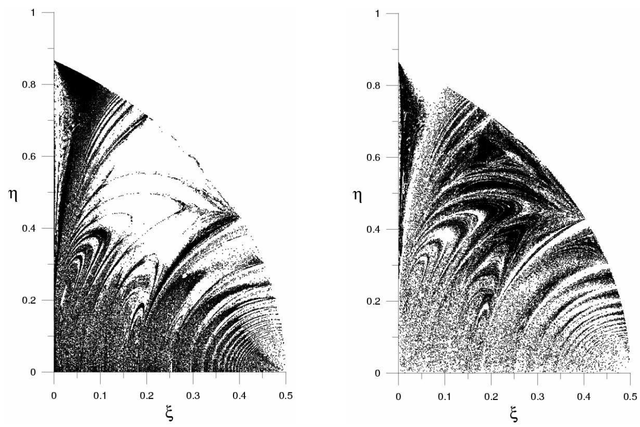

Fig. 6. Initial conditions corresponding to $0.69 < H_{1\,\max} < 0.70$ (left) and $1.09 < H_{1\,\max} < 1.10$ for triple encounters

dynamics. Some of those are given in [12]. Here we can add that distinct peaks on the entropies' histograms corresponding to different ergodic components could serve as one more experimental argument in favor of the applicability of the methods used. However we need to emphasize that the problem of proving the existence of a one-to-one correspondence between trajectories in the three-body problem and symbolic sequences is still open. It is the a subject of future studies.

Acknowledgments. A. Mylläri thanks the Magnus Ehrnrooths Foundation for the financial support. This work was also partly supported by the Leading Scientific Schools, Grants NSh-4929.2006.2 and NSh-5290.2006.2.

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Received October 4, 2006
Final Accepted October 18, 2007