Serdica J. Computing **1** (2007), 403–424

Serdica
Journal of Computing

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

A RELATION BETWEEN THE WEYL GROUP $W(E_8)$ AND EIGHT-LINE ARRANGEMENTS ON A REAL PROJECTIVE PLANE*

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ABSTRACT. The Weyl group $W(E_8)$ acts on the configuration space of systems of labelled eight lines on a real projective plane. With a system of eight lines with a certain condition, a diagram consisting of ten roots of the root system of type E_8 is associated. We have already shown the existence of a $W(E_8)$-equivariant map of the totality of such diagrams to the set of systems of labelled eight lines. The purpose of this paper is to report that the map is injective.

1. Introduction. We shall discuss simple eight-line arrangements on a real projective plane. Classifications of simple arrangements of six lines and seven

ACM Computing Classification System (1998): G.2.1
Key words: Weyl group, root system of type E_8, real projective plane, simple eight-line arrangement, classification of arrangement.

*The paper has been presented at the 12th International Conference on Applications of Computer Algebra, Varna, Bulgaria, June, 2006.

lines are well-known. In fact, it is proved by direct computation that there are eleven kinds of adjacent relations among polygons for seven-line arrangements (cf. [7]). This fact is in accord with what was described in Grünbaum's book [9], Chapter 18, namely, it is shown by Cummings and White (cf. [2], [3], [15]) that there are eleven different classes of non-equivalent seven-line arrangements in a real projective plane. The second author of this paper studied in detail the relationship between seven-line arrangements and the root system of type E_7 (cf. [11]). Let $\Delta(E_7)$ be the root system of type E_7. In Sekiguchi-Tanabata [13] (see also [11]), the notion of a tetrahedral set is introduced as that consisting of ten roots modulo sign in $\Delta(E_7)$ with a certain condition. Let $\mathcal{T}$ be the totality of tetrahedral sets. Then $W(E_7)$ acts on $\mathcal{T}$ in a natural manner. Let $\mathcal{P}_7$ be the set of connected components of seven-line configuration space. The following theorem is shown in [13], [11].

Theorem 1.

(i) *The set $\mathcal{T}$ is decomposed into fourteen S_7-orbits.*

(ii) *There is a $W(E_7)$-equivariant injective map of $\mathcal{T}$ to $\mathcal{P}_7$.*

By this theorem, we have fourteen S_7-orbits in $f(\mathcal{T})(\subset \mathcal{P}_7)$. These fourteen S_7-orbits are called types A, B1, B2, B3, B4, B5, C1, C2, C3, C4, D1, D2, D3, D4 (cf. [13], [11]). Among the seven-line arrangements of the fourteen types A, B1, ..., D4, the seven-line arrangements of type C2 and those of type D2 are equivalent and also the seven-line arrangements of type C4, those of type D1 and those of type D4 are equivalent. As a consequence, we find that seven-line arrangements of types C2 and D2 (or those of C4, D1, and D4) are not distinguished by adjacent relations among polygons and that there is a total of eleven kinds of seven-line arrangements from the systems of labelled seven lines of the fourteen types A, B1, ..., D4 distinguished by adjacent relations among polygons.

In this paper, we shall study a relationship between eight-line arrangements on a real projective plane and the root system of type E_8 on the same analogy of seven-line case. We have already studied simple eight-line arrangements in [4], [5], [6], [7], [8] and the references there. The theme treated in this paper is, among other things, related with the conjecture in [4].

The Weyl group $W(E_8)$ of type E_8, as will be defined in Section 2, acts on the configuration space of labelled eight lines with some conditions on a real projective plane. This configuration space is identified with an affine open subset

S of $\mathbf{R}^8$. Let $\mathcal{P}_8$ be the totality of connected components of S. Then $W(E_8)$ also acts on $\mathcal{P}_8$ (cf. §3). On the other hand, to each system of labelled eight lines, with some conditions, a diagram consisting of ten circles (roots in a root system of type E_8) analogous to a Dynkin diagram [4] is associated. Such the set and the diagram are called 8LC set and 8LC diagram, as will be introduced in Section 4. Let $\mathcal{LC}_8$ be the totality of 8LC sets. We have already shown [8] the existence of a $W(E_8)$-equivariant map f of $\mathcal{LC}_8$ to $\mathcal{P}_8$ (cf. §5).

The purpose of this paper is to report that the map f is injective. Our proof for this statement is deeply indebted to computation by computer and unfortunately not theoretical. At any rate, this implies that simple eight-line arrangements contained in connected components of $f(\mathcal{LC}_8) \subset \mathcal{P}_8$ are described in terms of the root system $\Delta(E_8)$. If f is surjective, the $W(E_8)$-structure of $\mathcal{P}_8$ is described in terms of the root system $\Delta(E_8)$ completely.

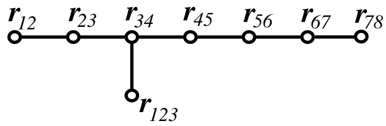
The first step in proving this statement is to determine all the representatives of S_8-orbits of $\mathcal{LC}_8$ by using symbolic computation. As a result, there are 2160 S_8-orbits of $\mathcal{LC}_8$. Let A_n $(n = 1, \dots, 2160)$ be the representatives of S_8-orbits. The second step to the proof is to determine $w_n \in W(E_8)$ satisfying $w_n \cdot \mathsf{U} = \mathsf{A}_n$ $(n = 2, \dots, 2160)$ where U is the S_8-orbit of the remarkable diagram described in our previous paper [8]. The first and second step will be explained in Section 6. The third step is to determine the labelled eight lines of $f(\mathsf{A}_n)$ by operating w_n on $f(\mathsf{U})$ successively. As a result, we conclude that systems of labelled eight lines contained in $f(\mathsf{A}_n)(n \neq 1)$ are not equivalent to those contained in $f(\mathsf{U})$ and the injectivity of f is proved in Section 7.

2. Root system of type E_8.

Let E be an 8-dimensional Euclidean space with an inner product $\langle \cdot, \cdot \rangle$ and an orthonormal basis $\{\boldsymbol{e}_j; 1 \le j \le 8\}$. We define the following 120 vectors of E:

$$
(1)\qquad
\begin{array}{lcll}
\boldsymbol{t}_1 & = & \dfrac{1}{2}\displaystyle\sum_{i=1}^{8} \boldsymbol{e}_i & \\
\boldsymbol{r}_{1j} & = & \boldsymbol{t}_1 - \boldsymbol{e}_{j-1} - \boldsymbol{e}_8 & (1 < j \le 8) \\
\boldsymbol{r}_{ij} & = & \boldsymbol{e}_{i-1} - \boldsymbol{e}_{j-1} & (1 < i < j \le 8) \\
\boldsymbol{r}_{1jk} & = & -\boldsymbol{e}_{j-1} - \boldsymbol{e}_{k-1} & (1 < j < k \le 8) \\
\boldsymbol{r}_{ijk} & = & \boldsymbol{t}_1 - \boldsymbol{e}_{i-1} - \boldsymbol{e}_{j-1} - \boldsymbol{e}_{k-1} - \boldsymbol{e}_8 & (1 < i < j < k \le 8) \\
\boldsymbol{t}_i & = & -\boldsymbol{e}_{i-1} - \boldsymbol{e}_8 & (1 < i \le 8) \\
\boldsymbol{t}_{1j} & = & \boldsymbol{e}_{j-1} - \boldsymbol{e}_8 & (1 < j \le 8) \\
\boldsymbol{t}_{ij} & = & \boldsymbol{t}_1 - \boldsymbol{e}_{i-1} - \boldsymbol{e}_{j-1} & (1 < i < j \le 8)
\end{array}
$$

The totality $\Delta(E_8)$ of vectors $\pm\boldsymbol{t}_i, \pm\boldsymbol{t}_{ij}, \pm\boldsymbol{r}_{ij}, \pm\boldsymbol{r}_{ijk}$ forms a root system of type E_8 [4]. It is clear that the set $\{\boldsymbol{r}_{12}, \boldsymbol{r}_{123}, \boldsymbol{r}_{23}, \boldsymbol{r}_{34}, \boldsymbol{r}_{45}, \boldsymbol{r}_{56}, \boldsymbol{r}_{67}, \boldsymbol{r}_{78}\}$ can serve as a system of positive roots; its Dynkin diagram is given as:



Let s_{ij}, s_{ijk} be the reflections on E with respect to $\boldsymbol{r}_{ij}, \boldsymbol{r}_{ijk}$ and let τ_i, τ_{ij} be the reflections on E with respect to $\boldsymbol{t}_i, \boldsymbol{t}_{ij}$ (cf. [4], [8]). We note here that the action of reflection s_{ij} $(i, j = 1, 2, \ldots, 8)$ causes transposition between the indices i and j on $\Delta(E_8)$. The group generated by the reflections $s_{ij}, s_{ijk}, \tau_i, \tau_{ij}$ is nothing but the Weyl group $W(E_8)$ of type E_8. In the sequel, the symmetric group S_8 is identified with the subgroup of $W(E_8)$ generated by s_{ij} unless otherwise stated.

3. Systems of labelled eight lines on a real projective plane.

In this section, we first introduce a $W(E_8)$-action on the set of systems of labelled eight lines on a real projective plane.

Let $(l_1, l_2, \ldots, l_8)$ be a system of labelled eight lines on $\mathbf{P}^2(\mathbf{R})$. We give conditions on $l_1, l_2, \ldots, l_8$:

I. The eight lines $l_1, l_2, \ldots, l_8$ are mutually different.

II. No three of $l_1, l_2, \ldots, l_8$ intersect at a point.

III. There is no conic tangent to any six of $l_1, l_2, \ldots, l_8$.

IV. Let $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_8$ be the dual points to $l_1, l_2, \ldots, l_8$. Then there is no cubic curve which passes through all of $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_8$ and which has a singularity at one of the eight points.

The system $(l_1, l_2, \ldots, l_8)$ defines a simple eight-line arrangement in the sense of Grünbaum [9] if the lines $l_1, l_2, \ldots, l_8$ satisfy the conditions I, II.

We define p-gons for the system of labelled eight lines $(l_j)_{1\le j\le 8}$. Each connected component of $\mathbf{P}^2(\mathbf{R}) - \cup_{j=1}^{8} l_j$ is called a polygon. If it is surrounded by p lines, it is called a p-gon.

The totality of systems of labelled eight lines on $\mathbf{P}^2(\mathbf{R})$ with conditions I, II forms the configuration space $\mathbf{P}(2,8)$; the space $\mathbf{P}(2,8)$ is defined by

$$\mathbf{P}(2,8) = GL(3,\mathbf{R})\backslash M'(3,8)/(\mathbf{R}^\times)^8,$$

where $M'(3,8)$ is the set of 3×8 real matrices of which no 3-minor vanishes. On the other hand, the totality of systems of labelled eight lines on $\mathbf{P}^2(\mathbf{R})$ with conditions I, II, III, IV forms a subset of $\mathbf{P}(2,8)$ which we denote by $\mathbf{P}_0(2,8)$. Both $\mathbf{P}(2,8)$ and $\mathbf{P}_0(2,8)$ are affine open subsets of $\mathbf{R}^8$. Permutations on the eight lines $l_1, l_2, \ldots, l_8$ induce a biregular S_8-action on $\mathbf{P}(2,8)$ (and also that on $\mathbf{P}_0(2,8)$). Let $\mathcal{P}_8$ be the set of connected components of $\mathbf{P}_0(2,8)$. It is stressed here that the S_8-action on $\mathbf{P}_0(2,8)$ is naturally extended to a birational $W(E_8)$-action (cf. [10], [12]). The $W(E_8)$-action on $\mathbf{P}_0(2,8)$ naturally induces that on $\mathcal{P}_8$.

We are going to define the action of $W(E_8)$ on $\mathbf{P}_0(2,8)$ in a concrete manner. Let $(l_j)_{1\le j\le 8}$ be a system of labelled eight lines. We assume that l_j is defined by

$$l_j : a_{1j}\xi + a_{2j}\eta + a_{3j}\zeta = 0, \tag{2}$$

where $(\xi : \eta : \zeta)$ is a homogeneous coordinate of $\mathbf{P}^2(\mathbf{R})$. For the system (l_j), we define a 3×8 matrix $X = (a_1, a_2, \ldots, a_8)$ where $a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{pmatrix}$.

By a projective linear transformation and scale ambiguity of (2), we may rewrite X to the following form

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & 1 & 1 & y_1 & y_2 & y_3 & y_4 \end{pmatrix}. \tag{3}$$

The matrix defined by (3) is called the normal form of X and written by $N(X)$ hereafter.

By the argument above, it is possible to choose as a representative of any element of $\mathbf{P}_0(2,8)$ a matrix of the form (3). Therefore $\mathbf{P}_0(2,8)$ is regarded as a quasi-affine subset of $\mathbf{R}^8$ by the correspondence

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & 1 & 1 & y_1 & y_2 & y_3 & y_4 \end{pmatrix} \longrightarrow (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4). \tag{4}$$

We introduce the following eight birational transformations $\sigma_1, \ldots, \sigma_7, \sigma_0$ on (x, y)-space (cf. [4]):

$$
\begin{array}{lll}
\sigma_1 : (x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) & \longrightarrow & \left(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3},\frac{1}{x_4},\frac{y_1}{x_1},\frac{y_2}{x_2},\frac{y_3}{x_3},\frac{y_4}{x_4}\right) \\
\sigma_2 : (x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) & \longrightarrow & (y_1,y_2,y_3,y_4,x_1,x_2,x_3,x_4) \\
\sigma_3 : (x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) & \longrightarrow & (x'_1,x'_2,x'_3,x'_4,y'_1,y'_2,y'_3,y'_4) \\
\sigma_4 : (x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) & \longrightarrow & \left(\frac{1}{x_1},\frac{x_2}{x_1},\frac{x_3}{x_1},\frac{x_4}{x_1},\frac{1}{y_1},\frac{y_2}{y_1},\frac{y_3}{y_1},\frac{y_4}{y_1}\right) \\
\sigma_5 : (x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) & \longrightarrow & (x_2,x_1,x_3,x_4,y_2,y_1,y_3,y_4) \\
\sigma_6 : (x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) & \longrightarrow & (x_1,x_3,x_2,x_4,y_1,y_3,y_2,y_4) \\
\sigma_7 : (x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) & \longrightarrow & (x_1,x_2,x_4,x_3,y_1,y_2,y_4,y_3) \\
\sigma_0 : (x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) & \longrightarrow & \left(\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3},\frac{1}{x_4},\frac{1}{y_1},\frac{1}{y_2},\frac{1}{y_3},\frac{1}{y_4}\right),
\end{array} \tag{5}
$$

where

$$x'_j = \frac{x_j - y_j}{1 - y_j}, \quad y'_j = \frac{y_j}{y_j - 1}, \quad j = 1,2,3,4.$$

Note that σ_j corresponds to the transposition of lines l_j and l_{j+1} $(1 \le j \le 8)$ and that the correspondence

$$s_{123} \longrightarrow \sigma_0, \quad s_{j-1,j} \longrightarrow \sigma_{j-1} \ (j = 2, \ldots, 8)$$

induces a surjective homomorphism $p_{W(E_8)}$ of $W(E_8)$ to the group $\tilde{W}(E_8)$ generated by $\sigma_0, \sigma_1, \ldots, \sigma_7$. In the sequel, we frequently identify $g \in W(E_8)$ with $p_{W(E_8)}(g)$ and subgroups of $W(E_8)$ with their images by $p_{W(E_8)}$ for simplicity.

We are now going to identify the space $\mathbf{P}_0(2,8)$ with a subset of $\mathbf{R}^8$ precisely. Let $X = (v_1v_2v_3v_4v_5v_6v_7v_8)$ be the matrix defined in (3). First put $R_{ijk} = \det(v_iv_jv_k)$ $(1 \le i < j < k \le 8)$. Clearly R_{123}, R_{124}, R_{134}, R_{23k} $(k = 4,5,6,7,8)$ are constants but the remaining R_{ijk} are polynomials of x, y. Moreover we take the polynomials T_{ij} $(1 \le i < j \le 8)$ and T_j $(1 \le j \le 8)$ defined in [8].

Then the following lemma holds (cf. [8]).

Lemma 1. *Let $l_1, l_2, \ldots, l_8$ be the lines defined from X in* (3).

(1) *If $R_{ijk} \neq 0$ for all i,j,k, then the condition I and II are satisfied.*

(2) *If $T_{ij} \neq 0$ for all i, j, then the condition III is satisfied.*

(3) *If $T_j \neq 0$ for all j, then the condition IV is satisfied.*

In virtue of Lemma 1, we find that the set $\mathbf{P}_0(2,8)$ is identified with the set

$$\{(x,y) \in \mathbf{R}^4 \times \mathbf{R}^4;\ D_{R,T}(x,y) \neq 0\},$$

where $D_{R,T}(x,y)$ is the product of all the polynomials R_{ijk}, T_{ij}, T_j. It is clear that $W(E_8)$ acts on $\mathbf{P}_0(2,8)$ biregularly.

4. 8LC sets and 8LC diagrams for the root system of type E_8. In next section, we will explain a relationship between the configuration space $\mathbf{P}_0(2,8)$ and the root system of type E_8. For this purpose, we first introduce the notions of 8LC sets and 8LC diagrams for the root system of type E_8.

Definition 1 (cf. [4])**.** *Let $a_i (i = 1, 2, \ldots, 8)$ and b_1, b_2 be roots of $\Delta(E_8)$. Then the set*

$$\mathsf{A} = \{a_i; i = 1, 2, \ldots, 8\} \cup \{b_1, b_2\} \tag{6}$$

is called an 8LC(= 8 lines configuration) set if the following conditions hold:

$$\begin{array}{lll} \text{(i)} & \langle a_i, a_j \rangle \neq 0 & \textit{if and only if } i - j \equiv 0 \textit{ or } \pm 1 \bmod 8. \\ \text{(ii)} & \langle b_1, b_2 \rangle = 0. & \\ \text{(iii.1)} & \langle a_i, b_1 \rangle \neq 0 & \textit{if and only if } i = 1. \\ \text{(iii.2)} & \langle a_i, b_2 \rangle \neq 0 & \textit{if and only if } i = 5. \end{array} \tag{7}$$

We would like to visualize each 8LC set by associating a diagram (analogous to a Dynkin diagram). Let $\mathsf{A} = \{a_i; i = 1, \ldots, 8\} \cup \{b_1, b_2\}$ be an 8LC set. Then an 8LC diagram for A is a figure consisting of ten circles attached with roots of A and segments constructed in Figure 1.

For an 8LC set $\mathsf{A} = \{a_i; i = 1, \ldots, 8\} \cup \{b_1, b_2\}$, we put

$$\tilde{\mathsf{A}} = \{\pm a_i; i = 1, \ldots, 8\} \cup \{\pm b_1, \pm b_2\} \tag{8}$$

and call it an extended 8LC set. Let A' be also an 8LC set. Then A and A' are *equivalent* if and only if $\tilde{\mathsf{A}} = \tilde{\mathsf{A}}'$. In this case, we always identify an 8LC diagram for A and that for A' for simplicity.

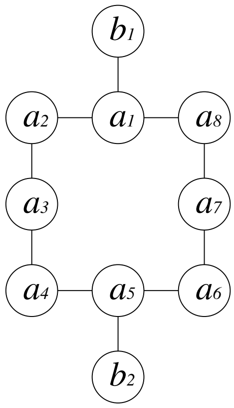


Fig. 1. 8LC diagram

The following lemma is shown by a direct computation.

Lemma 2 (cf. [8])**.**
If an 8LC set A *contains* $\{\boldsymbol{r}_{12},\ \boldsymbol{r}_{123},\ \boldsymbol{r}_{23},\ \boldsymbol{r}_{34},\ \boldsymbol{r}_{45},\ \boldsymbol{r}_{56},\ \boldsymbol{r}_{67},\ \boldsymbol{r}_{78}\}$ *(these form a set of simple roots of* $\Delta(E_8)$*), then* $\tilde{\mathsf{A}}$ *coincides with*

$$\{\pm\boldsymbol{r}_{12},\pm\boldsymbol{r}_{123},\pm\boldsymbol{r}_{23},\pm\boldsymbol{r}_{34},\pm\boldsymbol{r}_{45},\pm\boldsymbol{r}_{56},\pm\boldsymbol{r}_{67},\pm\boldsymbol{r}_{78},\pm\boldsymbol{t}_{18},\pm\boldsymbol{t}_{8}\}. \tag{9}$$

In virtue of this lemma, the classification of 8LC sets is essentially reduced to that of fundamental systems of roots of $\Delta(E_8)$ and this is well-known. Hence we get

Proposition 1. *Let* A *and* A' *be 8LC sets. Then there exists* $w \in W(E_8)$ *such that* $w \cdot \tilde{\mathsf{A}} = \tilde{\mathsf{A}}'$.

Let $\mathcal{LC}_8$ be the set of extended 8LC sets. We have already shown the existence of a $W(E_8)$-equivariant map f of $\mathcal{LC}_8$ to $\mathcal{P}_8$. The purpose of this paper is to show that the map f is injective.

5. The map of $\mathcal{LC}_8$ to $\mathcal{P}_8$.

In this section, we discuss the relationship between $\mathcal{LC}_8$ and $\mathcal{P}_8$. For this purpose, we consider the system of labelled eight lines $(l_1^0, l_2^0, \ldots, l_8^0)$ defined by the 3×8 matrix

$$X_0 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -\frac{27}{10} & -9 & \frac{3}{5} & -\frac{17}{10} & -\frac{3}{2} \\ 1 & 0 & 1 & 12 & 4 & \frac{9}{5} & \frac{53}{10} & \frac{21}{10} \end{pmatrix}. \tag{10}$$

This system $(l_1^0, l_2^0, \cdots, l_8^0)$ is remarkable in the sense that there is no hexagon for any system of labelled six lines constructed from $(l_1^0, l_2^0, \ldots, l_8^0)$ by taking off two lines and then it is clear that the system of $(l_1^0, l_2^0, \ldots, l_8^0)$ satisfy with all the conditions I, II, III, and IV. We denote by AE_8 the system of labelled eight lines $(l_1^0, l_2^0, \ldots, l_8^0)$ which is illustrated by Figure 2.

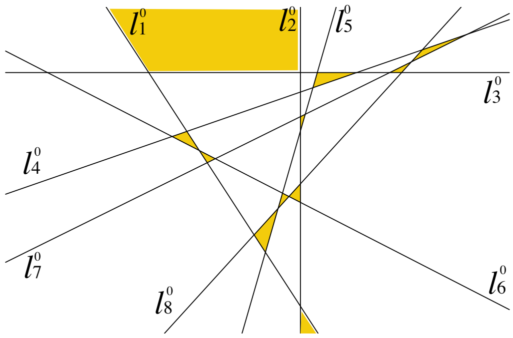


Fig. 2. The remarkable system of labelled eight lines

From the eight lines in Figure 2, we obtain ten triangles (Trn_k) $(k = 1, 2, \ldots, 10)$ surrounded by the three lines given in Table 1. We consider correspondence of ten triangles (Trn_k) $(k = 1, 2, \ldots, 10)$ to ten roots in Table 1.

Remark 1. The set $\mathsf{U} = \{\boldsymbol{r}_{123}, \boldsymbol{r}_{146}, \boldsymbol{r}_{158}, \boldsymbol{r}_{167}, \boldsymbol{r}_{257}, \boldsymbol{r}_{268}, \boldsymbol{r}_{345}, \boldsymbol{r}_{378}, \boldsymbol{r}_{478}, \boldsymbol{r}_{568}\}$ corresponding to triangles (Trn_k) $(k = 1, 2, \ldots, 10)$ in Table 1 is an 8LC set. In particular, the correspondence

$$\begin{array}{llllllllllll} \boldsymbol{r}_{568} & \longrightarrow & a_1 & \boldsymbol{r}_{268} & \longrightarrow & a_2 & \boldsymbol{r}_{345} & \longrightarrow & a_3 & \boldsymbol{r}_{167} & \longrightarrow & a_4 \\ \boldsymbol{r}_{146} & \longrightarrow & a_5 & \boldsymbol{r}_{378} & \longrightarrow & a_6 & \boldsymbol{r}_{478} & \longrightarrow & a_7 & \boldsymbol{r}_{123} & \longrightarrow & a_8 \\ \boldsymbol{r}_{158} & \longrightarrow & b_1 & \boldsymbol{r}_{257} & \longrightarrow & b_2 & & & & & & \end{array}$$

induces an 8LC diagram for U.

Put

$$g_1 = s_{16}s_{38}s_{57}\tau_{24}, \quad g_2 = s_{18}s_{27}s_{45}\tau_{36}, \quad g_3 = s_{23}s_{123}s_{45}s_{145}s_{67}s_{167}\tau_8\tau_{18}. \tag{11}$$

Table 1. Ten triangles

(Trn_1)	$l_1^0 l_2^0 l_3^0$	$\boldsymbol{r}_{123}$
(Trn_2)	$l_1^0 l_4^0 l_6^0$	$\boldsymbol{r}_{146}$
(Trn_3)	$l_1^0 l_5^0 l_8^0$	$\boldsymbol{r}_{158}$
(Trn_4)	$l_1^0 l_6^0 l_7^0$	$\boldsymbol{r}_{167}$
(Trn_5)	$l_2^0 l_5^0 l_7^0$	$\boldsymbol{r}_{257}$
(Trn_6)	$l_2^0 l_6^0 l_8^0$	$\boldsymbol{r}_{268}$
(Trn_7)	$l_3^0 l_4^0 l_5^0$	$\boldsymbol{r}_{345}$
(Trn_8)	$l_3^0 l_7^0 l_8^0$	$\boldsymbol{r}_{378}$
(Trn_9)	$l_4^0 l_7^0 l_8^0$	$\boldsymbol{r}_{478}$
(Trn_{10})	$l_5^0 l_6^0 l_8^0$	$\boldsymbol{r}_{568}$

Then g_1, g_2, g_3 generate the isotropy subgroup $Iso_{W(E_8)}(\tilde{\mathsf{U}})$ of $\tilde{\mathsf{U}}$ in $W(E_8)$, where $\tilde{\mathsf{U}}$ is the extended 8LC set of U. In particular, $Iso_{W(E_8)}(\tilde{\mathsf{U}}) \simeq (\mathbf{Z}_2)^3$. Note that g_3 is the generator of the center of $W(E_8)$.

Let C_{AE_8} be the connected component of $\mathcal{P}_8$ containing AE_8. In the paper [8], we have proved that any $g \in Iso_{W(E_8)}(\mathsf{U})$ leaves the set C_{AE_8} invariant, namely, $g_j \cdot C_{AE_8} = C_{AE_8}$ $(j = 1, 2, 3)$, where g_j is defined in (11). As a consequence, we have the following theorem.

Theorem 2 (cf. [8])**.** *Let f be the map of $\mathcal{LC}_8$ to $\mathcal{P}_8$ defined by $f(g \cdot \mathsf{U}) = g \cdot C_{AE_8}$. Then f is a $W(E_8)$-equivariant map of $\mathcal{LC}_8$ to $W(E_8) \cdot C_{AE_8}$.*

6. S_8-orbits of the totality of 8LC sets. In order to show that the $W(E_8)$-equivariant map f of $\mathcal{LC}_8$ to $\mathcal{P}_8$ is injective, we determine all the representatives of S_8-orbits of $\mathcal{LC}_8$.

Lemma 3. *There are* 2160 *S_8-orbits of $\mathcal{LC}_8$.*

Outline of proof. We explain the algorithm employed here to determine all the representatives of S_8-orbits of $\mathcal{LC}_8$. Let

$$\boldsymbol{R} = (R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9, R_{10}) \tag{12}$$

be a row vector consisting of roots of Δ_+ as in (1) such that $\{R_1, R_2, \ldots, R_{10}\}$ is an 8LC set by the correspondence

$$(13)\qquad \begin{array}{llllllllllll} R_1 & \longrightarrow & a_1 & R_2 & \longrightarrow & a_2 & R_3 & \longrightarrow & a_3 & R_4 & \longrightarrow & a_4 \\ R_5 & \longrightarrow & a_5 & R_6 & \longrightarrow & a_6 & R_7 & \longrightarrow & a_7 & R_8 & \longrightarrow & a_8 \\ R_9 & \longrightarrow & b_1 & R_{10} & \longrightarrow & b_2. & & & & & & \end{array}$$

We first introduce an ordering on the totality of such row vectors as $\boldsymbol{R}$. We number all the positive roots of Δ_+ in the following manner:

$$(14)\qquad \begin{aligned} &(R[1], R[2], \ldots, R[120]) \\ &= (\boldsymbol{t}_1, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_8, \boldsymbol{t}_{12}, \boldsymbol{t}_{13}, \ldots, \boldsymbol{t}_{78}, \boldsymbol{r}_{12}, \boldsymbol{r}_{13}, \ldots, \boldsymbol{r}_{78}, \boldsymbol{r}_{123}, \boldsymbol{r}_{124}, \ldots, \boldsymbol{r}_{678}). \end{aligned}$$

For example, $R[9] = \boldsymbol{t}_{12}$, $R[65] = \boldsymbol{r}_{123}$. We denote by $n(r)$ the number of a positive root r such that $r = R[n(r)]$ by (14).

Let $(n_1, n_2, \ldots, n_{10})$ and $(n'_1, n'_2, \ldots, n'_{10})$ be 10-row vectors consisting of integers. Then we define

$$(n_1, n_2, \ldots, n_{10}) \prec (n'_1, n'_2, \ldots, n'_{10}),$$

if and only if $n_j = n'_j$ $(1 \le j \le k-1)$, $n_k < n'_k$ and $(n_1, n_2, \ldots, n_{10}) = (n'_1, n'_2, \ldots, n'_{10})$ if and only if $n_i = n'_i$ $(i = 1, \ldots, 10)$. Let $\boldsymbol{R} = (R_1, R_2, \ldots, R_{10})$, $\boldsymbol{R}' = (R'_1, R'_2, \ldots, R'_{10})$ be row vectors consisting of ten roots. Then $\boldsymbol{R} \prec \boldsymbol{R}'$ if and only if

$$\left(n(R_1), n(R_2), \ldots, n(R_{10})\right) \prec \left(n(R'_1), n(R'_2), \ldots, n(R'_{10})\right).$$

In this way, we define an order in the set of row vectors as $\boldsymbol{R}$.

Let $\mathcal{U}$ be an S_8-orbit of $\mathcal{LC}_8$. Then we take $\boldsymbol{R} = (R_1, R_2, \ldots, R_{10})$ with the conditions

(1) $\{R_1, R_2, \ldots, R_{10}\} \in \mathcal{U}$.

(2) If $\tilde{\mathsf{A}} = \{\pm R'_1, \pm R'_2, \ldots, \pm R'_{10}\}$ is any element of $\mathcal{U}$ $(R'_i \in \Delta_+)$ then

$$\boldsymbol{R} \preceq (R'_1, R'_2, \ldots, R'_{10}).$$

It is not obvious whether the row vector $\boldsymbol{R}$ is uniquely determined by $\mathcal{U}$ or not. (In fact, we will explain later that there are four candidates of it.) At any rate, we write the row vector $\boldsymbol{R}$ defined above for $\boldsymbol{R}(\mathcal{U})$ for a moment.

With the help of computer algebra system Mathematica, we determine the row vector $\boldsymbol{R}(\mathcal{U})$ for any S_8-orbit $\mathcal{U}$ of $\mathcal{LC}_8$.

At this moment, we need a comment. Let $\mathsf{A} = \{R_1, R_2, \ldots, R_{10}\}$ be an 8LC set (cf. (13)) and let $\mathcal{U}$ be the S_8-orbit of A. Then there are four different row vectors

$$\begin{aligned} \boldsymbol{R}^{(1)} &= (R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9, R_{10}), \\ \boldsymbol{R}^{(2)} &= (R_1, R_8, R_7, R_6, R_5, R_4, R_3, R_2, R_9, R_{10}), \\ \boldsymbol{R}^{(3)} &= (R_5, R_4, R_3, R_2, R_1, R_8, R_7, R_6, R_{10}, R_9), \\ \boldsymbol{R}^{(4)} &= (R_5, R_6, R_7, R_8, R_1, R_2, R_3, R_4, R_{10}, R_9). \end{aligned} \tag{15}$$

These come from the symmetry of up and down and that of left and right of 8LC diagram. Possibly the S_8-orbits of $\boldsymbol{R}^{(1)}, \boldsymbol{R}^{(2)}, \boldsymbol{R}^{(3)}, \boldsymbol{R}^{(4)}$ are different in spite that the corresponding 8LC set is A. This is the reason why $\boldsymbol{R}(\mathcal{U})$ is not uniquely determined by $\mathcal{U}$. Moreover, $\boldsymbol{R}^{(1)}, \boldsymbol{R}^{(2)}, \boldsymbol{R}^{(3)}, \boldsymbol{R}^{(4)}$ are identified in the course of our computation.

As a result, we conclude that there exist 2160 S_8-orbits $\mathcal{U}_n$ ($n = 1, \ldots,$ 2160). Actually we obtained 4×2160 row vectors which are of form $\boldsymbol{R}(\mathcal{U})$. Only some of 2160 representatives are given in Table 2 since all the data is very huge. The first column in Table 2 stands for the classified number of the S_8-orbits and second column stands for the concrete representative of the S_8-orbit of $\mathcal{LC}_8$. $\square$

Remark 2. We do not know an efficient method constructing S_8-orbits of $\mathcal{LC}_8$ other than exhaustive trial method. It is convenient for the calculation to use the following property. Since S_8 acts on $\mathcal{U}$ and since S_8 acts on the sets $\{\boldsymbol{t}_1, \ldots, \boldsymbol{t}_8\}$, $\{\boldsymbol{t}_{12}, \ldots, \boldsymbol{t}_{78}\}$, $\{\boldsymbol{r}_{12}, \ldots, \boldsymbol{r}_{78}\}$, $\{\boldsymbol{r}_{123}, \ldots, \boldsymbol{r}_{678}\}$ transitively, we may take as R_1 one of $\boldsymbol{t}_1$, $\boldsymbol{t}_{12}$, $\boldsymbol{r}_{12}$, $\boldsymbol{r}_{123}$. In virtue of (13), the roots R_2, R_8, and R_9 are not orthogonal to R_1 and the remaining six roots are. The total of possible combination is $4 \times {}_{63}P_9 \simeq 3.4 \times 10^{16}$ way. The computational complexity becomes $\dfrac{{}_{63}P_9}{{}_{120}P_{10}} \simeq \dfrac{1}{49040}$ times.

Remark 3. Since the order of Weyl group $W(E_8)$ is equal to $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ and since the isotropy of U (cf. (16)) in $W(E_8)$ is isomorphic to ${\mathbf{Z}_2}^3$, we observe that there are at least $\dfrac{|W(E_8)|}{|S_8| \times |{\mathbf{Z}_2}^3|} = 2160$ number of S_8-orbits of $\mathcal{P}_8$ (cf. [5]) . This actually coincides with the total number of $\mathcal{LC}_8$ just as we have obtained in Lemma 3. As a consequence, we find that for any 8LC set A, $\mathrm{Aut}_{W(E_8)}(\mathsf{A}) \cap S_8 = \{1\}$, where $\mathrm{Aut}_{W(E_8)}(\mathsf{A}) = \{w \in W(E_8) | w\mathsf{A} = \mathsf{A}\}$.

Table 2. Some representatives of 2160 S_8-orbits of $\mathcal{LC}_8$

n	Representative A_n of S_8-orbit $\mathcal{U}_n$	$\mathsf{A}_n = w \cdot \mathsf{A}_i$, $(w \in W(E_8))$
1	$\{\boldsymbol{r}_{123}, \boldsymbol{r}_{124}, \boldsymbol{r}_{356}, \boldsymbol{r}_{178}, \boldsymbol{r}_{157}, \boldsymbol{r}_{268}, \boldsymbol{r}_{258}, \boldsymbol{r}_{467}, \boldsymbol{r}_{237}, \boldsymbol{r}_{348}\}$	1
2	$\{\boldsymbol{r}_{12}, \boldsymbol{r}_{134}, \boldsymbol{r}_{567}, \boldsymbol{r}_{128}, \boldsymbol{r}_{125}, \boldsymbol{r}_{368}, \boldsymbol{r}_{358}, \boldsymbol{r}_{246}, \boldsymbol{r}_{237}, \boldsymbol{r}_{478}\}$	$s_{37}s_{27}s_{237}\mathsf{A}_1$
3	$\{\boldsymbol{t}_{12}, \boldsymbol{r}_{123}, \boldsymbol{r}_{134}, \boldsymbol{r}_{256}, \boldsymbol{r}_{178}, \boldsymbol{r}_{157}, \boldsymbol{r}_{248}, \boldsymbol{r}_{367}, \boldsymbol{r}_{358}, \boldsymbol{r}_{168}\}$	$s_{78}s_{68}s_{48}s_{12}s_{248}\mathsf{A}_{141}$
4	$\{\boldsymbol{t}_{12}, \boldsymbol{r}_{345}, \boldsymbol{r}_{167}, \boldsymbol{r}_{136}, \boldsymbol{r}_{138}, \boldsymbol{r}_{148}, \boldsymbol{r}_{237}, \boldsymbol{r}_{568}, \boldsymbol{r}_{125}, \boldsymbol{r}_{246}\}$	$s_{45}s_{67}s_{35}s_{78}s_{58}s_{28}s_{14}s_{148}\mathsf{A}_{901}$
5	$\{\boldsymbol{r}_{123}, \boldsymbol{r}_{124}, \boldsymbol{t}_{14}, \boldsymbol{r}_{256}, \boldsymbol{r}_{156}, \boldsymbol{r}_{157}, \boldsymbol{r}_{346}, \boldsymbol{r}_{467}, \boldsymbol{r}_{458}, \boldsymbol{r}_{168}\}$	$s_{78}\tau_{28}\mathsf{A}_6$
6	$\{\boldsymbol{r}_{123}, \boldsymbol{r}_{124}, \boldsymbol{t}_{14}, \boldsymbol{r}_{256}, \boldsymbol{r}_{347}, \boldsymbol{r}_{158}, \boldsymbol{r}_{157}, \boldsymbol{r}_{468}, \boldsymbol{r}_{136}, \boldsymbol{r}_{345}\}$	$s_{56}s_{78}s_{68}s_{48}s_{38}s_{24}s_{248}\mathsf{A}_{30}$
7	$\{\boldsymbol{r}_{123}, \boldsymbol{r}_{124}, \boldsymbol{t}_{14}, \boldsymbol{r}_{256}, \boldsymbol{r}_{347}, \boldsymbol{r}_{345}, \boldsymbol{r}_{167}, \boldsymbol{r}_{136}, \boldsymbol{r}_{468}, \boldsymbol{r}_{158}\}$	$s_{56}\tau_{27}\mathsf{A}_6$
8	$\{\boldsymbol{r}_{123}, \boldsymbol{r}_{124}, \boldsymbol{t}_{35}, \boldsymbol{r}_{167}, \boldsymbol{r}_{348}, \boldsymbol{r}_{346}, \boldsymbol{r}_{158}, \boldsymbol{r}_{237}, \boldsymbol{r}_{457}, \boldsymbol{r}_{256}\}$	$s_{67}s_{78}s_{58}s_{48}s_{12}s_{13}s_{38}s_{25}s_{14}\tau_{67}\mathsf{A}_6$
…	…	…
30	$\{\boldsymbol{r}_{12}, \boldsymbol{r}_{134}, \boldsymbol{r}_{567}, \boldsymbol{r}_{58}, \boldsymbol{r}_{368}, \boldsymbol{r}_{127}, \boldsymbol{r}_{126}, \boldsymbol{t}_{16}, \boldsymbol{r}_{158}, \boldsymbol{r}_{34}\}$	$s_{67}s_{57}s_{78}s_{38}s_{12}s_{18}s_{138}\mathsf{A}_1$
31	$\{\boldsymbol{r}_{123}, \boldsymbol{r}_{14}, \boldsymbol{r}_{156}, \boldsymbol{r}_{57}, \boldsymbol{r}_{267}, \boldsymbol{r}_{23}, \boldsymbol{r}_{368}, \boldsymbol{t}_{68}, \boldsymbol{r}_{578}, \boldsymbol{r}_{148}\}$	$s_{56}s_{67}s_{23}s_{17}s_{137}\mathsf{A}_{20}$
32	$\{\boldsymbol{r}_{123}, \boldsymbol{r}_{14}, \boldsymbol{r}_{245}, \boldsymbol{r}_{56}, \boldsymbol{r}_{357}, \boldsymbol{r}_{356}, \boldsymbol{r}_{278}, \boldsymbol{r}_{134}, \boldsymbol{t}_{23}, \boldsymbol{r}_{78}\}$	$s_{78}s_{67}\tau_{27}\mathsf{A}_{35}$
33	$\{\boldsymbol{r}_{123}, \boldsymbol{r}_{14}, \boldsymbol{t}_{15}, \boldsymbol{r}_{145}, \boldsymbol{r}_{146}, \boldsymbol{r}_{67}, \boldsymbol{r}_{256}, \boldsymbol{r}_{567}, \boldsymbol{r}_{38}, \boldsymbol{r}_{358}\}$	$s_{78}s_{58}s_{23}s_{367}\mathsf{A}_{22}$
34	$\{\boldsymbol{r}_{123}, \boldsymbol{r}_{14}, \boldsymbol{t}_{15}, \boldsymbol{r}_{145}, \boldsymbol{r}_{146}, \boldsymbol{r}_{257}, \boldsymbol{r}_{256}, \boldsymbol{r}_{27}, \boldsymbol{r}_{38}, \boldsymbol{r}_{358}\}$	$s_{78}s_{68}s_{58}s_{47}s_{247}\mathsf{A}_1$
…	…	…
134	$\{\boldsymbol{r}_{123}, \boldsymbol{t}_{45}, \boldsymbol{r}_{46}, \boldsymbol{r}_{156}, \boldsymbol{r}_{157}, \boldsymbol{r}_{257}, \boldsymbol{r}_{258}, \boldsymbol{r}_{38}, \boldsymbol{t}_{12}, \boldsymbol{r}_{246}\}$	$s_{78}s_{68}s_{45}s_{58}s_{23}\tau_{18}\mathsf{A}_{34}$
…	…	…
141	$\{\boldsymbol{t}_{12}, \boldsymbol{r}_{123}, \boldsymbol{r}_{34}, \boldsymbol{r}_{145}, \boldsymbol{r}_{267}, \boldsymbol{r}_{256}, \boldsymbol{r}_{178}, \boldsymbol{r}_{346}, \boldsymbol{t}_{16}, \boldsymbol{r}_{78}\}$	$s_{78}s_{58}s_{45}s_{12}\tau_{34}\mathsf{A}_{540}$
…	…	…
540	$\{\boldsymbol{t}_{12}, \boldsymbol{r}_{123}, \boldsymbol{t}_{45}, \boldsymbol{r}_{167}, \boldsymbol{r}_{146}, \boldsymbol{r}_{257}, \boldsymbol{r}_{247}, \boldsymbol{r}_{356}, \boldsymbol{t}_{26}, \boldsymbol{t}_{37}\}$	$s_{67}s_{45}s_{68}s_{46}s_{34}\tau_3\mathsf{A}_1$
…	…	…
901	$\{\boldsymbol{t}_1, \boldsymbol{r}_{123}, \boldsymbol{r}_{456}, \boldsymbol{r}_{245}, \boldsymbol{r}_{247}, \boldsymbol{r}_{347}, \boldsymbol{r}_{268}, \boldsymbol{r}_{157}, \boldsymbol{r}_{148}, \boldsymbol{r}_{358}\}$	$s_{56}s_{47}s_{38}\tau_{18}\mathsf{A}_{1981}$
…	…	…
1321	$\{\boldsymbol{r}_{12}, \boldsymbol{t}_{13}, \boldsymbol{r}_{34}, \boldsymbol{r}_{45}, \boldsymbol{r}_{56}, \boldsymbol{r}_{67}, \boldsymbol{r}_{78}, \boldsymbol{r}_{28}, \boldsymbol{t}_1, \boldsymbol{r}_{345}\}$	$s_{67}s_{46}s_{34}s_{12}s_{25}s_{13}s_{125}\mathsf{A}_{134}$
…	…	…
1981	$\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{r}_{234}, \boldsymbol{r}_{345}, \boldsymbol{r}_{356}, \boldsymbol{r}_{478}, \boldsymbol{r}_{467}, \boldsymbol{r}_{146}, \boldsymbol{r}_{137}, \boldsymbol{r}_{368}\}$	$s_{78}s_{67}s_{57}s_{48}s_{37}s_{24}s_{13}\tau_{12}\mathsf{A}_1$
…	…	…
2152	$\{\boldsymbol{t}_{12}, \boldsymbol{t}_{13}, \boldsymbol{r}_{245}, \boldsymbol{r}_{46}, \boldsymbol{r}_{67}, \boldsymbol{r}_{146}, \boldsymbol{t}_1, \boldsymbol{t}_3, \boldsymbol{r}_{25}, \boldsymbol{r}_{78}\}$	$s_{78}\tau_{47}\mathsf{A}_{644}$
2153	$\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_{13}, \boldsymbol{r}_{456}, \boldsymbol{r}_{47}, \boldsymbol{r}_{78}, \boldsymbol{r}_{358}, \boldsymbol{r}_{13}, \boldsymbol{t}_{25}, \boldsymbol{r}_{46}\}$	$s_{46}s_{68}s_{38}s_{567}\mathsf{A}_{2116}$
2154	$\{\boldsymbol{t}_1, \boldsymbol{t}_{23}, \boldsymbol{t}_{12}, \boldsymbol{r}_{345}, \boldsymbol{r}_{46}, \boldsymbol{r}_{67}, \boldsymbol{r}_{78}, \boldsymbol{r}_{138}, \boldsymbol{t}_2, \boldsymbol{r}_{45}\}$	$s_{45}s_{56}s_{26}s_{16}s_{126}\mathsf{A}_{1801}$
2155	$\{\boldsymbol{t}_{12}, \boldsymbol{r}_{345}, \boldsymbol{r}_{36}, \boldsymbol{r}_{34}, \boldsymbol{r}_{45}, \boldsymbol{t}_{57}, \boldsymbol{t}_1, \boldsymbol{t}_7, \boldsymbol{r}_{28}, \boldsymbol{r}_{258}\}$	$s_{45}s_{34}s_{36}s_{12}s_{13}\tau_1\mathsf{A}_{983}$
2156	$\{\boldsymbol{t}_1, \boldsymbol{r}_{123}, \boldsymbol{r}_{24}, \boldsymbol{r}_{45}, \boldsymbol{r}_{56}, \boldsymbol{r}_{67}, \boldsymbol{t}_{17}, \boldsymbol{t}_8, \boldsymbol{t}_{38}, \boldsymbol{r}_{367}\}$	$\tau_1\mathsf{A}_{1723}$
2157	$\{\boldsymbol{t}_{12}, \boldsymbol{t}_3, \boldsymbol{t}_1, \boldsymbol{t}_{34}, \boldsymbol{r}_{45}, \boldsymbol{r}_{246}, \boldsymbol{r}_{67}, \boldsymbol{r}_{26}, \boldsymbol{r}_{458}, \boldsymbol{r}_{58}\}$	$s_{67}s_{56}s_{145}\mathsf{A}_{1783}$
2158	$\{\boldsymbol{t}_1, \boldsymbol{t}_{23}, \boldsymbol{r}_{24}, \boldsymbol{r}_{256}, \boldsymbol{r}_{57}, \boldsymbol{r}_{56}, \boldsymbol{t}_{16}, \boldsymbol{t}_3, \boldsymbol{r}_{124}, \boldsymbol{r}_{78}\}$	$s_{12}\tau_1\mathsf{A}_{1936}$
2159	$\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_{13}, \boldsymbol{r}_{456}, \boldsymbol{r}_{47}, \boldsymbol{r}_{45}, \boldsymbol{r}_{56}, \boldsymbol{t}_{26}, \boldsymbol{r}_{13}, \boldsymbol{r}_{78}\}$	$s_{56}s_{46}s_{57}s_{35}s_{23}s_{14}s_{124}\mathsf{A}_{1324}$
2160	$\{\boldsymbol{t}_{12}, \boldsymbol{t}_3, \boldsymbol{t}_1, \boldsymbol{r}_{145}, \boldsymbol{r}_{46}, \boldsymbol{r}_{67}, \boldsymbol{r}_{78}, \boldsymbol{r}_{28}, \boldsymbol{t}_{13}, \boldsymbol{r}_{45}\}$	$s_{34}s_{12}s_{13}\tau_{13}\mathsf{A}_{710}$

Example 1. *Let $\mathcal{U}_1$ be the S_8-orbit of $\mathcal{LC}_8$ containing*

$$\mathsf{U} = \{\boldsymbol{r}_{568}, \boldsymbol{r}_{268}, \boldsymbol{r}_{345}, \boldsymbol{r}_{167}, \boldsymbol{r}_{146}, \boldsymbol{r}_{378}, \boldsymbol{r}_{478}, \boldsymbol{r}_{123}, \boldsymbol{r}_{158}, \boldsymbol{r}_{257}\} \tag{16}$$

given in Remark 1. *In this case, it is easy to see that*

$$\boldsymbol{R}(\mathcal{U}_1) = (\boldsymbol{r}_{123}, \boldsymbol{r}_{124}, \boldsymbol{r}_{356}, \boldsymbol{r}_{178}, \boldsymbol{r}_{157}, \boldsymbol{r}_{268}, \boldsymbol{r}_{258}, \boldsymbol{r}_{467}, \boldsymbol{r}_{237}, \boldsymbol{r}_{348}) \tag{17}$$

and

$$\mathsf{A}_1 = s_0 \cdot \mathsf{U} \tag{18}$$

where $s_0 = s_{78}s_{56}s_{58}s_{46}s_{12}s_{13}s_{38}s_{26}s_{15}$. Let X_0 be the matrix defined in (10) *and let $N(X_0)$ be its normal form, namely*

$$N(X_0) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & a'_1 & a'_2 & a'_3 & a'_4 \\ 0 & 0 & 1 & 1 & b'_1 & b'_2 & b'_3 & b'_4 \end{pmatrix}. \tag{19}$$

We put $(a, b) = s_0(a', b')$. Then by direct computation, we find that

$$\begin{aligned} a &= \left(\frac{28709}{26389}, \frac{304}{275}, \frac{646}{385}, \frac{27113}{25982}\right), \\ b &= \left(\frac{4313}{2399}, \frac{133}{75}, \frac{494}{175}, \frac{2109}{1181}\right). \end{aligned} \tag{20}$$

Let X_1 be the normal form of 3×8 matrix corresponding to (a, b). Then it follows from the definition that X_1 is contained in $f(\mathsf{A}_1)$.

Since we have shown in Lemma 3 that there are 2160 S_8-orbits of $\mathcal{LC}_8$, we denote by $\mathcal{U}_n$ $(n = 1, \ldots, 2160)$ these S_8-orbits. For each $\mathcal{U}_n$, we have defined $\boldsymbol{R}(\mathcal{U}_n)$. We denote by A_n the 8LC set defined by the row vector $\boldsymbol{R}(\mathcal{U}_n)$. In spite that $\boldsymbol{R}(\mathcal{U}_n)$ is not uniquely determined by $\mathcal{U}_n$, A_n is uniquely determined. (In our computation, for the technical reason A_n is determined ahead. Then $\mathcal{U}_n$ is done. This is not essential.) We choose $\mathcal{U}_1$ so that $\mathsf{U} \in \mathcal{U}_1$ (cf. Example 1).

To continue the computation, we choose and fix $w_n \in W(E_8)$ $(n = 2, \ldots, 2160)$ satisfying

$$w_n \cdot \mathsf{A}_1 = \mathsf{A}_n. \tag{21}$$

Some of concrete forms of w_n $(n = 1, \ldots, 2160)$ are given in the last column in Table 2.

Example 2. *Note that the representative* A_{1321} *includes simple roots (Dynkin diagram). Then the 8LC set (9) in Lemma 2 is contained in* $\mathcal{U}_{1321}$*. In Table 2, we can see the relation between* A_1 *and* A_{1321} *as follows:*

$$\begin{aligned} \mathsf{A}_{34} &= s_{78}s_{68}s_{58}s_{47}s_{247}\mathsf{A}_1 \\ \mathsf{A}_{134} &= s_{78}s_{68}s_{45}s_{58}s_{23}\tau_{18}\mathsf{A}_{34} \\ \mathsf{A}_{1321} &= s_{67}s_{46}s_{34}s_{12}s_{25}s_{13}s_{125}\mathsf{A}_{134}. \end{aligned} \tag{22}$$

7. The main theorem. In this section, we will prove the injectivity of the map f defined in Theorem 2. Let X be a matrix of the form (3) and let $(x,y) = (x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4)$ be the point of $\boldsymbol{R}^8$ defined by X. Then $C(X)$ denotes the connected component of $\mathcal{P}_8$ containing (x,y). Clearly, $C(N(X_0)) = C_{AE_8}$ where X_0 is defined by (10).

We choose 3×8 matrices X_n $(n=1,\ldots,2160)$ satisfying $f(\mathsf{A}_n) = C(X_n)$ in the following way. We take X_1 as the one defined in Example 1 and also the point (a,b) of $\boldsymbol{R}^8$ in Example 1. On the other hand, we already chose w_n in (21). Then we put

$$(a^{(1)},b^{(1)}) = (a,b),\ (a^{(n)},b^{(n)}) = w_n\cdot(a,b)\ (n=2,\ldots,2160) \tag{23}$$

and let X_n be the matrix corresponding to $(a^{(n)},b^{(n)})$. Then it follows from the definition of X_n and A_n that $f(\mathsf{A}_n) = C(X_n)$.

Before stating the next lemma, we recall the definition of the adjacent relation among polygons (cf. [6], [7]). Let $\mathcal{A}(H)$ be an n-line arrangement, where $H=(l_1,l_2,\ldots,l_n)$. There are $M = \dfrac{n(n-1)}{2}+1$ number of polygons in $\mathcal{A}(H)$. We denote all of polygons by Σ_j $(j=1,2,\ldots,M)$. If Σ_j is a p-gon, there are p number of polygons $\Sigma_{j_1},\ldots,\Sigma_{j_p}$ having common side with Σ_j. If Σ_{j_k} is an N_{j_k}-gon $(k=1,\ldots,p)$, we put $R_{\Sigma_j} = \{N_{j_1},\ldots,N_{j_p}\}$. We may assume that $N_1\le N_2\le\ldots\le N_p$. We call R_{Σ_j} *the list of adjacent polygons for the p-gon* Σ_j.

Definition 2. *We denote the totality of the lists of adjacent polygons for all polygons* $\Sigma_1,\Sigma_2,\ldots,\Sigma_M$ *in* $\mathcal{A}(H)$ *by* $R(\mathcal{A}(H)) = \{R_{\Sigma_1},R_{\Sigma_2},\ldots,R_{\Sigma_M}\}$ *and call* $R(\mathcal{A}(H))$ *the adjacent relation among polygons in an n-line arrangement* $\mathcal{A}(H)$ *which may be sorted in lexicographical order.*

We return to our situation. Let $\mathcal{A}_n$ be the simple eight-line arrangement defined by the system of eight lines corresponding to the matrix X_n of (23) in the sense of Grünbaum [9] $(n=1,\ldots,2160)$.

By using the symbolic computational system Mathematica, we show the following lemma.

Lemma 4.

$$S_8 \cdot C(X_n) \neq S_8 \cdot C(X_1) \quad (n = 2, \ldots, 2160). \tag{24}$$

Outlie of proof. It is sufficient to show that $\mathcal{A}_n \neq \mathcal{A}_1$ $(n \geq 2)$.

Let $R(\mathcal{A}_n)$ $(n = 1, \ldots, 2160)$ be the adjacent relation among polygons in $\mathcal{A}_n$. In particular, we give $R(\mathcal{A}_1)$. In this case, there are ten triangles, thirteen squares and six pentagons in $\mathcal{A}_1$. Its adjacent relation among polygons is given by

$$\begin{aligned} R(\mathcal{A}_1) \;=\; & \{\{4,4,4\},\{4,4,5\},\{4,4,5\},\{4,4,5\},\{4,4,5\}, \\ & \{4,4,5\},\{4,5,5\},\{4,5,5\},\{4,5,5\},\{5,5,5\}, \\ & \{3,3,4,5\},\{3,3,4,5\},\{3,3,4,5\},\{3,4,4,4\}, \\ & \{3,4,4,5\},\{3,4,4,5\},\{3,4,4,5\},\{3,4,4,5\},\{3,4,4,5\}, \\ & \{3,4,5,5\},\{3,4,5,5\},\{3,4,5,5\},\{3,4,5,5\}, \\ & \{3,3,3,4,4\},\{3,3,3,4,4\},\{3,3,4,4,4\}, \\ & \{3,3,4,4,4\},\{3,3,4,4,4\},\{3,3,4,4,4\}\}. \end{aligned} \tag{25}$$

By direct computation using Mathematica, we have determined all the adjacent relation among polygons $R(\mathcal{A}_n)$. The totality of $R(\mathcal{A}_n)$ $(n = 1, \ldots, 2160)$ is divided into 135 different families of the adjacent relations among polygons. Our result is shown in Table 3. The first column $R(\mathcal{A})$ in Table 3 stands for classified number $1, \ldots, 135$ of $R(\mathcal{A}_n)$ $(n = 1, \ldots, 2160)$ and the second column for numbers of (8,7,6,5,4,3-gon) in the arrangement with $R(\mathcal{A})$. The third column F stands for the number of arrangements within the family with $R(\mathcal{A})$ and the fourth column $\mathcal{A}_n$ with $R(\mathcal{A})$ stands for such the F number of arrangements.

Looking at Table 3, we conclude that the family $R(\mathcal{A}) = 20$ contains only $\mathcal{A}_1$. This means that

$$R(\mathcal{A}_n) \neq R(\mathcal{A}_1) \quad (n = 2, \ldots, 2160).$$

Then,

$$\mathcal{A}_n \neq \mathcal{A}_1 \quad (n = 2, \ldots, 2160). \tag{26}$$

Table 3. The adjacent relation among polygons of $\mathcal{A}_n$ $(n = 1, \ldots, 2160)$

$R(\mathcal{A})$	8,7,6,5,4,3-gon	$^*F$	$^{**}\mathcal{A}_n$ with $R(\mathcal{A})$
1	0,0,0,4,17,8	1	$\mathcal{A}_{26}$
2		6	$\mathcal{A}_{29}, \mathcal{A}_{35}, \ldots$
3		12	$\mathcal{A}_{208}, \mathcal{A}_{977}, \ldots$
4		24	$\mathcal{A}_{186}, \mathcal{A}_{410}, \ldots$
5	0,0,0,5,15,9	1	$\mathcal{A}_{2}$
6		6	$\mathcal{A}_{22}, \mathcal{A}_{947}, \ldots$
7		3	$\mathcal{A}_{140}, \mathcal{A}_{1144}, \mathcal{A}_{1527}$
8		3	$\mathcal{A}_{19}, \mathcal{A}_{937}, \mathcal{A}_{1091}$
9		10	$\mathcal{A}_{181}, \mathcal{A}_{339}, \ldots$
10		4	$\mathcal{A}_{24}, \mathcal{A}_{25}, \ldots$
11		8	$\mathcal{A}_{111}, \mathcal{A}_{112}, \ldots$
12		4	$\mathcal{A}_{18}, \mathcal{A}_{55}, \ldots$
13		12	$\mathcal{A}_{162}, \mathcal{A}_{163}, \ldots$
14		12	$\mathcal{A}_{28}, \mathcal{A}_{148}, \ldots$
15		10	$\mathcal{A}_{104}, \mathcal{A}_{105}, \ldots$
16		8	$\mathcal{A}_{54}, \mathcal{A}_{938}, \ldots$
17		16	$\mathcal{A}_{32}, \mathcal{A}_{169}, \ldots$
18		14	$\mathcal{A}_{482}, \mathcal{A}_{731}, \ldots$
19	0,0,0,6,13,10	3	$\mathcal{A}_{13}, \mathcal{A}_{928}, \mathcal{A}_{1040}$
20*		1	$\mathcal{A}_{1}$
21		5	$\mathcal{A}_{16}, \mathcal{A}_{927}, \ldots$
22		5	$\mathcal{A}_{10}, \mathcal{A}_{918}, \ldots$
23		14	$\mathcal{A}_{130}, \mathcal{A}_{135}, \ldots$
24		2	$\mathcal{A}_{51}, \mathcal{A}_{52}$
25		7	$\mathcal{A}_{101}, \mathcal{A}_{931}, \ldots$
26		14	$\mathcal{A}_{132}, \mathcal{A}_{295}, \ldots$
27		16	$\mathcal{A}_{113}, \mathcal{A}_{119}, \ldots$
28		12	$\mathcal{A}_{23}, \mathcal{A}_{98}, \ldots$
29		12	$\mathcal{A}_{444}, \mathcal{A}_{456}, \ldots$
30		9	$\mathcal{A}_{102}, \mathcal{A}_{247}, \ldots$
31		14	$\mathcal{A}_{121}, \mathcal{A}_{251}, \ldots$
32		11	$\mathcal{A}_{144}, \mathcal{A}_{398}, \ldots$
33	0,0,0,7,11,11	3	$\mathcal{A}_{4}, \mathcal{A}_{910}, \mathcal{A}_{998}$
34		5	$\mathcal{A}_{6}, \mathcal{A}_{909}, \ldots$
35		7	$\mathcal{A}_{65}, \mathcal{A}_{922}, \ldots$
36		4	$\mathcal{A}_{908}, \mathcal{A}_{988}, \ldots$
37		1	$\mathcal{A}_{8}$
38		7	$\mathcal{A}_{62}, \mathcal{A}_{915}, \ldots$
39		10	$\mathcal{A}_{17}, \mathcal{A}_{72}, \ldots$
40		9	$\mathcal{A}_{58}, \mathcal{A}_{218}, \ldots$
41		14	$\mathcal{A}_{268}, \mathcal{A}_{269}, \ldots$
42		26	$\mathcal{A}_{106}, \mathcal{A}_{142}, \ldots$
43	0,0,0,8,9,12	7	$\mathcal{A}_{40}, \mathcal{A}_{904}, \ldots$
44		5	$\mathcal{A}_{45}, \mathcal{A}_{46}, \ldots$
45		6	$\mathcal{A}_{50}, \mathcal{A}_{239}, \ldots$
46		14	$\mathcal{A}_{90}, \mathcal{A}_{91}, \ldots$
47		2	$\mathcal{A}_{555}, \mathcal{A}_{556}$
48	0,0,0,9,7,13	5	$\mathcal{A}_{227}, \mathcal{A}_{228}, \ldots$
49	0,0,1,2,18,8	20	$\mathcal{A}_{36}, \mathcal{A}_{156}, \ldots$
50		26	$\mathcal{A}_{211}, \mathcal{A}_{478}, \ldots$
51		11	$\mathcal{A}_{427}, \mathcal{A}_{428}, \ldots$
52		56	$\mathcal{A}_{187}, \mathcal{A}_{194}, \ldots$
53	0,0,1,3,16,9	10	$\mathcal{A}_{30}, \mathcal{A}_{34}, \ldots$
54		10	$\mathcal{A}_{31}, \mathcal{A}_{124}, \ldots$
55		16	$\mathcal{A}_{27}, \mathcal{A}_{174}, \ldots$
56		14	$\mathcal{A}_{93}, \mathcal{A}_{151}, \ldots$
57		22	$\mathcal{A}_{131}, \mathcal{A}_{176}, \ldots$
58		38	$\mathcal{A}_{196}, \mathcal{A}_{197}, \ldots$
59		38	$\mathcal{A}_{188}, \mathcal{A}_{409}, \ldots$
60		6	$\mathcal{A}_{125}, \mathcal{A}_{959}, \ldots$
61		16	$\mathcal{A}_{33}, \mathcal{A}_{116}, \ldots$
62		24	$\mathcal{A}_{206}, \mathcal{A}_{209}, \ldots$
63		18	$\mathcal{A}_{92}, \mathcal{A}_{157}, \ldots$
64		40	$\mathcal{A}_{155}, \mathcal{A}_{159}, \ldots$
65		40	$\mathcal{A}_{191}, \mathcal{A}_{192}, \ldots$
66		30	$\mathcal{A}_{175}, \mathcal{A}_{184}, \ldots$
67	0,0,1,4,14,10	7	$\mathcal{A}_{21}, \mathcal{A}_{108}, \ldots$
68		7	$\mathcal{A}_{114}, \mathcal{A}_{279}, \ldots$
69		14	$\mathcal{A}_{69}, \mathcal{A}_{126}, \ldots$
70		20	$\mathcal{A}_{180}, \mathcal{A}_{182}, \ldots$
71		22	$\mathcal{A}_{171}, \mathcal{A}_{172}, \ldots$
72		18	$\mathcal{A}_{165}, \mathcal{A}_{344}, \ldots$
73		20	$\mathcal{A}_{167}, \mathcal{A}_{173}, \ldots$
74		9	$\mathcal{A}_{20}, \mathcal{A}_{133}, \ldots$
75		15	$\mathcal{A}_{161}, \mathcal{A}_{328}, \ldots$
76		20	$\mathcal{A}_{168}, \mathcal{A}_{367}, \ldots$
77		14	$\mathcal{A}_{453}, \mathcal{A}_{454}, \ldots$
78		34	$\mathcal{A}_{153}, \mathcal{A}_{351}, \ldots$
79		46	$\mathcal{A}_{143}, \mathcal{A}_{146}, \ldots$
80		46	$\mathcal{A}_{115}, \mathcal{A}_{120}, \ldots$
81	0,0,1,5,12,11	6	$\mathcal{A}_{14}, \mathcal{A}_{74}, \ldots$
82		14	$\mathcal{A}_{471}, \mathcal{A}_{674}, \ldots$
83		10	$\mathcal{A}_{15}, \mathcal{A}_{926}, \ldots$
84		4	$\mathcal{A}_{11}, \mathcal{A}_{12}, \ldots$
85		7	$\mathcal{A}_{9}, \mathcal{A}_{78}, \ldots$
86		12	$\mathcal{A}_{84}, \mathcal{A}_{85}, \ldots$
87		14	$\mathcal{A}_{75}, \mathcal{A}_{88}, \ldots$
88		13	$\mathcal{A}_{96}, \mathcal{A}_{243}, \ldots$
89		14	$\mathcal{A}_{79}, \mathcal{A}_{86}, \ldots$
90		20	$\mathcal{A}_{118}, \mathcal{A}_{271}, \ldots$
91	0,0,1,6,10,12	4	$\mathcal{A}_{3}, \mathcal{A}_{5}, \ldots$
92		9	$\mathcal{A}_{56}, \mathcal{A}_{59}, \ldots$
93		11	$\mathcal{A}_{64}, \mathcal{A}_{923}, \ldots$
94		17	$\mathcal{A}_{66}, \mathcal{A}_{306}, \ldots$
95		14	$\mathcal{A}_{95}, \mathcal{A}_{261}, \ldots$
96		3	$\mathcal{A}_{7}, \mathcal{A}_{991}, \mathcal{A}_{1005}$
97		11	$\mathcal{A}_{57}, \mathcal{A}_{70}, \ldots$
98		9	$\mathcal{A}_{63}, \mathcal{A}_{67}, \ldots$
99		20	$\mathcal{A}_{82}, \mathcal{A}_{302}, \ldots$
100	0,0,1,7,8,13	3	$\mathcal{A}_{43}, \mathcal{A}_{999}, \mathcal{A}_{1336}$
101		5	$\mathcal{A}_{38}, \mathcal{A}_{41}, \ldots$
102		9	$\mathcal{A}_{39}, \mathcal{A}_{48}, \ldots$
103		13	$\mathcal{A}_{44}, \mathcal{A}_{230}, \ldots$
104		6	$\mathcal{A}_{266}, \mathcal{A}_{267}, \ldots$
105		16	$\mathcal{A}_{262}, \mathcal{A}_{263}, \ldots$
106	0,0,1,8,6,14	6	$\mathcal{A}_{225}, \mathcal{A}_{226}, \ldots$
107	0,0,2,0,19,8	24	$\mathcal{A}_{517}, \mathcal{A}_{522}, \ldots$
108	0,0,2,1,17,9	56	$\mathcal{A}_{214}, \mathcal{A}_{499}, \ldots$
109		24	$\mathcal{A}_{520}, \mathcal{A}_{521}, \ldots$
110	0,0,2,2,15,10	16	$\mathcal{A}_{487}, \mathcal{A}_{488}, \ldots$
111		22	$\mathcal{A}_{189}, \mathcal{A}_{407}, \ldots$
112		18	$\mathcal{A}_{412}, \mathcal{A}_{435}, \ldots$
113	0,0,2,3,13,11	12	$\mathcal{A}_{441}, \mathcal{A}_{442}, \ldots$
114	0,0,2,4,11,12	5	$\mathcal{A}_{700}, \mathcal{A}_{701}, \ldots$
115		21	$\mathcal{A}_{100}, \mathcal{A}_{352}, \ldots$
116		16	$\mathcal{A}_{389}, \mathcal{A}_{390}, \ldots$
117		19	$\mathcal{A}_{97}, \mathcal{A}_{99}, \ldots$
118		12	$\mathcal{A}_{395}, \mathcal{A}_{396}, \ldots$
119		29	$\mathcal{A}_{103}, \mathcal{A}_{354}, \ldots$
120	0,0,2,5,9,13	16	$\mathcal{A}_{312}, \mathcal{A}_{313}, \ldots$
121		19	$\mathcal{A}_{61}, \mathcal{A}_{217}, \ldots$
122		13	$\mathcal{A}_{341}, \mathcal{A}_{558}, \ldots$
123	0,0,2,6,7,14	5	$\mathcal{A}_{236}, \mathcal{A}_{237}, \ldots$
124		7	$\mathcal{A}_{37}, \mathcal{A}_{902}, \ldots$
125	0,0,2,7,5,15	6	$\mathcal{A}_{223}, \mathcal{A}_{224}, \ldots$
126	0,0,3,6,4,16	5	$\mathcal{A}_{544}, \mathcal{A}_{545}, \ldots$
127	0,1,0,1,19,8	96	$\mathcal{A}_{210}, \mathcal{A}_{215}, \ldots$
128	0,1,0,2,17,9	48	$\mathcal{A}_{190}, \mathcal{A}_{195}, \ldots$
129		114	$\mathcal{A}_{185}, \mathcal{A}_{199}, \ldots$
130	0,1,0,3,15,10	24	$\mathcal{A}_{149}, \mathcal{A}_{160}, \ldots$
131	0,1,0,5,11,12	18	$\mathcal{A}_{53}, \mathcal{A}_{134}, \ldots$
132		22	$\mathcal{A}_{107}, \mathcal{A}_{128}, \ldots$
133		28	$\mathcal{A}_{94}, \mathcal{A}_{123}, \ldots$
134	0,1,0,7,7,14	2	$\mathcal{A}_{987}, \mathcal{A}_{1001}$
135	1,0,0,0,20,8	62	$\mathcal{A}_{518}, \mathcal{A}_{519}, \ldots,$ $\mathcal{A}_{1321}, \ldots, \mathcal{A}_{2160}$

*The column F stands for the number of arrangements within the family with $R(\mathcal{A})$.

**The column $\mathcal{A}_n$ with $R(\mathcal{A})$ stands for the members of arrangements with $R(\mathcal{A})$ but elements more than the third are omitted.

Hence we conclude that $S_8 \cdot C(X_n) \neq S_8 \cdot C(X_1)$ and the lemma follows. □

We are in a position to prove the main theorem.

Theorem 3. *The map f of $\mathcal{LC}_8$ to $W(E_8) \cdot C_{AE_8}$ is injective.*

Proof. Let $\tilde{\mathsf{A}}, \tilde{\mathsf{A}}'$ be extended 8LC sets and assume that $f(\tilde{\mathsf{A}}) = f(\tilde{\mathsf{A}}')$. Then it suffices to show that $\tilde{\mathsf{A}} = \tilde{\mathsf{A}}'$.

Since the action of $W(E_8)$ on $\mathcal{LC}_8$ is transitive, we may assume that $\mathsf{A} = \mathsf{U}$ without loss of generality, where U is the 8LC set introduced in Example 1.

First treat the case $\tilde{\mathsf{A}}' \in \mathcal{U}_n$ for some n (> 1). Then we find from Lemma 4 that $f(\tilde{\mathsf{A}}) \neq f(\tilde{\mathsf{A}}')$. This contradicts the assumption.

Next treat the case $\tilde{\mathsf{A}}' \in \mathcal{U}_1$, namely there is $w \in W(E_8)$ such that $w \cdot \mathsf{U} = \mathsf{A}'$. Then by the assumption, $f(\tilde{\mathsf{U}}) = f(\tilde{\mathsf{A}}') = w \cdot f(\tilde{\mathsf{U}})$. It suffices to show that $w = 1$. To do so, we examine the relation between the eight lines and ten triangles for the system of labelled eight lines $(l_1^0, l_2^0, \ldots, l_8^0)$ corresponding to the matrix $N(X_0)$. We find the following properties from Table 1:

(1) l_8^0 is sides of five triangles $l_1^0 l_5^0 l_8^0$, $l_2^0 l_6^0 l_8^0$, $l_3^0 l_7^0 l_8^0$, $l_4^0 l_7^0 l_8^0$ and $l_5^0 l_6^0 l_8^0$.

(2) l_1^0, l_5^0, l_6^0, and l_7^0 are sides of four triangles respectively.

(3) l_2^0, l_3^0, and l_4^0 are sides of three triangles respectively.

(4) $l_1^0 l_2^0 l_3^0$ is the unique triangle which has l_1^0 and l_2^0 as sides.

From these properties, we first observe that l_8^0 plays a role different from the remaining seven lines. Comparing **(2)** and the triangles containing lines l_6^0 and l_7^0 of the five triangles in **(1)**, we find that the roles of l_6^0 and l_7^0 are different from the remaining lines. Then from the remaining two triangles $l_1^0 l_5^0 l_8^0$ and $l_5^0 l_6^0 l_8^0$ we also find that the roles of l_1^0 and l_5^0 are different from the remaining lines. At this moment, we remark that the roles of l_1^0 and l_2^0 are different from the others. From the remark and the property **(4)**, we find that the role of l_3^0 is different from the remaining lines. In this way, we conclude that eight lines play different roles each other.

Now we put $(x', y') = w \cdot (a', b')$ where (a', b') is defined in Example 1 and let X' be the matrix of the normal form corresponding to (x', y'). Then X' defines a system of labelled eight lines $(l_1', l_2', \ldots, l_8')$. From the assumption, (x', y') is contained in $f(\tilde{\mathsf{U}})$. This implies that $(l_1^0, \ldots, l_8^0)$ is continuously deformed to $(l_1', \ldots, l_8')$ preserving the conditions I, II, III, IV. On the other hand, $l_1', \ldots, l_8'$ have the same properties **(1)**, **(2)**, **(3)**, **(4)**. As a consequence, l_8^0 becomes to l_8'.

Similarly $l_1^0, \dots, l_7^0$ become to $l_1', \dots, l_7'$, respectively. This implies that $w = 1$ and $w \cdot \tilde{\mathsf{A}} = \tilde{\mathsf{U}}$. Therefore the injectivity of f is completely proved. □

Remark 4. As for the case $\tilde{\mathsf{A}}' \in \mathcal{U}_1$ in the proof of Theorem 3, it is possible to imply the conclusion by an alternative argument, which we now explain.

For any $\mathsf{A}' = s \cdot \mathsf{U}, \ \ s \in S_8, \ s \neq 1$, it suffices to show that $f(\tilde{\mathsf{A}}') \neq f(\tilde{\mathsf{U}})$. To do so from Lemma 1, we compare the signs of R_{ijk} $(1 \leq i < j < k \leq 8)$ of (a, b) by (20) and those of $s \cdot (a, b)$. By direct computation, we find that the 48-vector;

$$\begin{aligned} &(\mathrm{Sign}(R_{125}(a,b)), \mathrm{Sign}(R_{126}(a,b)), \dots, \mathrm{Sign}(R_{678}(a,b))) \\ &= (1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,1,1,1,-1,-1,-1,1,1,1,1,1,-1, \\ &1,-1,-1,1,1,-1,-1,-1,-1,-1,-1,1,1,-1,-1,-1,-1,1,1,1,-1,1,1,-1) \end{aligned} \tag{27}$$

where Sign(n) gives $-1, 0, 1$ if $n < 0$, $n = 0$, $n > 0$, respectively. By direct computation using Mathematica, we obtain that the 48-vector (27) is different from others by $s \cdot (a, b), \ \ s \in S_8, \ s \neq 1$. Note here that there are $8! = 40320$ number of permutations in S_8. As a result, we find that $f(\tilde{\mathsf{A}}') \neq f(\tilde{\mathsf{A}})$. This contradict the assumption.

Conjecture 1. *The map f of $\mathcal{LC}_8$ to $\mathcal{P}_8$ is bijective.*

If this is true, any system of labelled eight lines is described in terms of root system of type E_8 and simple eight-line arrangements are completely classified.

Finally observing the result of $R(\mathcal{A}_n)$ $(n = 1, \dots, 2160)$ in the outline of proof of Lemma 4, we summarize the following proposition.

Proposition 2. *Eight-line arrangements obtained from systems of labelled eight lines contained in $f(\mathcal{LC}_8)$ are divided into* 28 *number of families by the difference of the numbers of polygons and are divided into* 135 *number of families by the difference of the adjacent relations among polygons (cf. Table* 3*).*

Remark 5. This study started from Problem 2 in [4] (see also Problem 2 in [8] p. 372). Then, the system of labelled eight lines corresponding to the matrix X_1 by (20) has a remarkable property among all the labelled eight lines from our point of view. We have obtained only two solutions to Problem 2 in [4]. The one solution is represented by the labelled eight lines corresponding to X_1

and the other is represented by X_2.

$$X_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \frac{4313}{2399} & \frac{133}{75} & \frac{494}{175} & \frac{2109}{1181} \\ 0 & 0 & 1 & 1 & \frac{54}{989} & \frac{3}{20} & -\frac{117}{70} & -\frac{33}{698} \end{pmatrix}. \tag{28}$$

The arrangement $\mathcal{A}_2$ of the system of labelled eight lines corresponding to X_2 belongs to the member of $R(\mathcal{A}) = 5$ in Table 3 and contains nine triangles, fifteen squares, and five pentagons.

Remark 6. Looking at Table 3, there are 62 members of $R(\mathcal{A}) = 135$. In this case, there is one octagon. Especially, we observe that the arrangement $\mathcal{A}_{1321}$ corresponding to Example 2 is contained in these members. Figure 3 is an illustration of eight lines of this family.

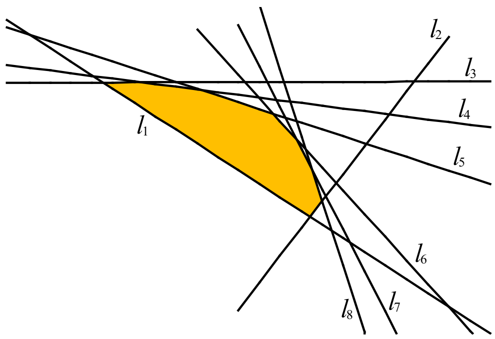


Fig. 3. Eight lines of $R(\mathcal{A}) = 135$

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Received October 4, 2006
Final Accepted October 18, 2007