Serdica J. Computing **1** (2007), 207–212

Serdica
Journal of Computing

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

IMPROVEMENTS ON THE JUXTAPOSING THEOREM

Igor Gashkov, Henrik Larsson

Abstract. A new class of binary constant weight codes is presented. We establish new lower bound and exact values on $A(n_1+n_2, 2(a_1+a_2), n_2) \geq \min\{M_1, M_2\}+1$, if $A(n_1, 2a_1, a_1+b_1) = M_1$ and $A(n_2, 2b_2, a_2+b_2) = M_2$, in particular, $A(30, 16, 15) = 16$ and $A(33, 18, 15) = 11$.

1. Introduction. Let $A(n, d, w)$ denote the maximum possible number of codewords of length n, minimum distance d apart and constant weight w. To find the maximum number of codewords of a code given the length, the minimum distance and a constant weight is one of the most basic problems in coding theory. There are both upper bounds and lower bounds on $A(n, d, w)$. Upper bounds are the maximum number or words that can be found in theory while lower bounds are given by actual code constructions. If these bounds are equal then the code is called *optimal*.

Tables of these constant weight codes are regularly updating, but there are still many question marks in these tables. A well-known paper by A. E. Brouwer, James B. Shearer and N. J. A Sloane [2] in 1990 gave an extensive

ACM Computing Classification System (1998): E.4.
Key words: Constant weight code, juxtaposing theorem, optimal code.

table of lower bounds on these codes and different ways of finding them. Most of the values in these tables are optimal, but there are some that can be improved on. There are also a lot of question marks in the tables, which means that there exists no known theorem that can construct a good code for these parameters. There is also an online version of the previously mentioned paper on the Internet: `http://www.research.att.com/~njas/codes/Andw` [4]. The following theorems [see [2]) give some lower bounds on $A(n, d, w)$.

Theorem 1 (basic values). *Given $A(n, d, w)$ we have:*

$A(n, d, w) = A(n, d+1, w)$, *whenever d is odd.*

$A(n, d, w) = A(n, d, n-w)$.

$A(n, d, w) = 1$, *if* $2w < d$.

$A(n, 2w, w) = \left\lfloor \frac{n}{w} \right\rfloor$.

$A(n, 2, w) = \left\{ \begin{matrix} n \\ w \end{matrix} \right\}$.

Theorem 2 (juxtaposing theorem). *If we have two codes C_1 and C_2 with $M_1 = A(n_1, d_1, w_1)$ and $M_2 = A(n_2, d_2, w_2)$ number of codewords, we can* juxtapose *these codes (place them side by side) and obtain a new code $C_3 = (n_1 + n_2, d_1 + d_2, w_1 + w_2)$ that has $M_3 = A(n_1 + n_2, d_1 + d_2, w_1 + w_2) \geq \min\{M_1, M_2\}$ codewords.*

This theorem states that we can actually fill the holes in the tables of constant weight codes with juxtaposed codes [4]. In other words any question mark can be replaced with a new code constructed by two smaller codes (although the juxtaposing theorem does not often give good results with larger codes).

If one can find a lower bound for a constant weight code that is equal to the theoretical upper bound then that code is an optimal code. Agrell, Vardy and Zeger wrote an important paper on upper bounds in June 2000 [1]. Upper bounds are calculated with the help of theorems for upper bounds for constant weight codes or by using combinations of these theorems. Upper bounds do not help to construct a code but rather finding the maximum number of codewords that can be found.

Theorem 3 (upper bound). $A(n, 2\delta, w) \leq \left\lfloor \frac{\delta * n}{w^2 - wn + \delta n} \right\rfloor$, *if the denominator is positive.*

There are many different theorems for upper bounds and this is only one of them. Theorem 3 will be used later to show that the upper bound for $A(30, 16, 15)$ is 16.

2. New theorems and new optimal codes. Given two codes C_1 and C_2 we can place them side-by-side and obtain a new code (by using the juxtaposing theorem). If we have $A(n_1, d_1, w_1)$ number of codewords in C_1 and $A(n_2, d_2, w_2)$ number of codewords in C_2 we get the new code C_3 that has

$$A(n_1 + n_2, d_1 + d_2, w_1 + w_2) \geq \min\{A(n_1, d_1, w_1), A(n_2, d_2, w_2)\}$$

number of codewords. The question is if we can improve on this result. In other words – is there anyway to add more vectors to C_3 that share the same length, distance and weight? The answer is yes. In [3] we have new theorem on this subject, which gives some new lower bounds.

Theorem 4. *If* $A(n_1, 2a, a + b) = M_1$ *and* $A(n_2, 2b, a + b) = M_2$*, then* $A(n_1 + n_2, 2(a + b), \min\{n_1, n_2\}) \geq \min\{M_1, M_2\} + 1$.

We denote this new code as $(n_1, 2a, a + b) \cup (n_2, 2b, a + b)$.

With the help of Theorem 4 we can find yet another way of adding vectors to codes that has been placed side-by-side. The idea is similar in the way that we add a vector consisting of zeros in the first n_1 positions and ones in the last n_2 positions but is a greater generalization. The difference is that we do not restrict the new codes to be combined by smaller codes that are of the same weight. This gives us more options on how to construct them.

Theorem 5. *If* $A(n_1, 2a_1, a_1 + b_1) = M_1$ *and* $A(n_2, 2a_2, a_2 + b_2) = M_2$ *then,*

$$A(n_1 + n_2, 2(a_1 + a_2), n_2) \geq \min\{M_1, M_2\} + 1,$$

if the following two conditions are satisfied:

$$n_2 = a_1 + b_1 + a_2 + b_2, \quad b_1 \geq a_2.$$

We denote this new code as $(n_1, 2a_1, a_1 + b_1)\hat{\cup}(n_2, 2b_2, a_2 + b_2)$.

Proof. Let $n_1 \geq n_2$ and juxtapose the two codes $(n_1, 2a_1, a_1 + b_1)$ and $(n_2, 2a_2, a_2 + b_2)$. We obtain $(n_1 + n_2, 2(a_1 + a_2), a_1 + b_1 + a_2 + b_2)$. We can add a binary vector to this code consisting of zeros in the first n_1 positions and ones in the last n_2 positions, i.e., $v = (0 \;\; 0 \;\; \ldots \;\; 0 \;\; 1 \;\; \ldots \;\; 1 \;\; 1)$. Clearly this vector has the weight n_2. The distance between v and any other vector in the code is the sum of two distances. The first is the distance between the first n_1 positions of v and the vectors of the code $(n_1, 2a_1, a_1 + b_1)$. This distance is equal to the weight $a_1 + b_1$. The second distance is the distance between the last n_2 positions of v and the vectors of the code $(n_2, 2a_2, a_2 + b_2)$. This distance is $n_2 - a_2 - b_2$ and we have the total distance $a_1 + b_1 + n_2 - a_2 - b_2$. Since $n_2 = a_1 + b_1 + a_2 + b_2$ we can rewrite this distance as $a_1 + b_1 + a_1 + b_1 = 2a_1 + 2b_1$. We have $b_1 \geq a_2$ from the second condition of the theorem so $2(a_1 + b_1) \geq 2(a_1 + a_2)$.

Example. This example will use theorem 5 to find the optimal code for (30, 16, 15). It is the first non-trivial example using this theorem. We want to construct this code by combining the two smaller codes (15, 8, 8) and (15, 8, 7), i.e., $(15,8,8)\hat{\cup}(15,8,7)$. We have $n_1 = 15$, $a_1 = 4$, $b_1 = 4$ and $n_2 = 15$, $a_2 = 4$, $b_2 = 3$. From the tables of constant weight codes we know that that $A(15,8,8) = 15$ and $A(15,8,7) = 15$. Theorem 2 should then give us $A(30,16,15) = \min\{15,15\} + 1 = 15 + 1 = 16$, instead of the old value $A(30,16,15) = 15$ that is also found in the tables [4]. First we juxtapose $(15,8,8)$ and $(15,8,7)$:

$$\begin{array}{r@{\;}ccccccccccccccccccccccccccccccc}
 & \multicolumn{15}{c}{(15,8,8)} & \multicolumn{15}{c}{(15,8,7)} \\
v_1 = & 1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1 \\
v_2 = & 0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1 \\
v_3 = & 0&0&1&1&0&0&1&1&1&1&0&0&1&1&0&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1 \\
v_4 = & 1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&1&1&0&0&1&1&0&0&0&0&1&1&0&0&1 \\
v_5 = & 0&0&1&1&1&1&0&0&1&0&1&0&1&0&1&0&0&1&1&1&1&0&0&0&1&0&1&0&1&0 \\
v_6 = & 1&1&0&0&0&0&1&1&1&0&1&0&1&0&1&1&1&0&0&0&0&1&1&0&1&0&1&0&1&0 \\
v_7 = & 1&0&1&0&1&0&1&0&1&0&0&1&0&1&1&1&0&1&0&1&0&1&0&0&1&1&0&1&0&0 \\
v_8 = & 0&1&0&1&0&1&0&1&1&0&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&0 \\
v_9 = & 1&0&1&0&0&1&0&1&0&1&1&0&0&1&1&1&0&1&0&0&1&0&1&1&0&0&1&1&0&0 \\
v_{10} = & 0&1&0&1&1&0&1&0&0&1&1&0&0&1&1&0&1&0&1&1&0&1&0&1&0&0&1&1&0&0 \\
v_{11} = & 1&0&0&1&1&0&0&1&0&1&0&1&1&0&1&1&0&0&1&1&0&0&1&1&0&1&0&0&1&0 \\
v_{12} = & 0&1&1&0&0&1&1&0&0&1&0&1&1&0&1&0&1&1&0&0&1&1&0&1&0&1&0&0&1&0 \\
v_{13} = & 1&0&0&1&0&1&1&0&0&0&1&1&1&1&0&1&0&0&1&0&1&1&0&1&1&0&0&0&0&1 \\
v_{14} = & 0&1&1&0&1&0&0&1&0&0&1&1&1&1&0&0&1&1&0&1&0&0&1&1&1&0&0&0&0&1 \\
v_{15} = & 1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1.
\end{array}$$

And with the help of Theorem 5 we can add the vector:

$$\begin{array}{r@{\;}cccccccccccccccccccccccccccccc}
v_{16} = & 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\,.
\end{array}$$

It is easy to see that v_{16} is of length 30, distance 16 and weight 15. The weight is obvious as it is equal to the length of $(15,8,7)$. The distance is the sum of the distance between a zero vector and $(15, 8, 8)$, and the distance between an all one vector and $(15, 8, 7)$. This sum is $8 + 15 - 7 = 16$.

We have found a new code. With the help of theorem 5 we have added one vector to $(30, 16, 15)$. Now the obvious question is if there are more vectors that we haven't found yet? In this case the answer is no. With the help of theorem 3 we can see that the upper bound for $(30, 16, 15)$ is in fact 16.

$$A(30, 2 * 8, 15) \leq \left\lfloor \frac{8 * 30}{15^2 - 15 * 30 + 8 * 30} \right\rfloor = 16$$

This means that we have found the optimal number of code vectors that can be found for $(30, 16, 15)$, i.e., the *optimal code*. Using the same construction we have $A(33, 18, 15) = 11$ optimal code as $(18, 10, 9)\hat{\cup}(15, 8, 6)$.

By using Theorems 4 and 5 we can find many new codes (some of them are optimal, see example above and some are close to optimal codes see [3]) with a minimum distance d greater than 8. Some are poor while others are good. Theorem 4 and 5 are similar in many ways since they both also add one binary vector to two already juxtaposed codes. In fact, many of the codes that can be constructed using theorem 4 can also be constructed using Theorem 5. The opposite does not hold however since Theorem 5 do not restrict the two juxtaposed codes to be of the same weight. The presented theorems lead to several improvements of the tables of lower bounds on $A(n, d, w)$ maintained by E. M. Rains and N. J. A. Sloane [4], and the ones recently published by D. H. Smith, L. A. Hughes and S. Perkins [5].

REFERENCES

[1] Agrell E., A. Vardy, K. Zeger. Upper bounds for constant-weight codes. *IEEE Trans. Inform. Theory*, **46**, *7* (2000), 2373–2395.

[2] Brouwer E., J. B. Shearer, N. J. A. Sloane. A new table of Constant weight codes. *IEEE Trans. Inform. Theory*, **36**, *6* (1990), 1334–1380.

[3] Gashkov I. Constant Weight Codes with Package CodingTheory.m in Mathematica. Lecture Notes in Comput. Sci. vol. **3039**, 2004, 370–375

[4] Rains E. M., N. J. A. Sloane. Table of constant weight binary codes. `http://www.research.att.com/njas/codes/Andw/`.

[5] Smith D. H., L. A. Hughes, S. Perkins. A new table of constant weight codes of length greater than 28. *Electron. J. Combin.* **13** (2006), 1–18.

Karlstad University
Department of Engineering Sciences,
Physics and Mathematics
65188 Karlstad Sweden
e-mail: `Igor.Gachkov@kau.se`
`master_henrik@hotmail.com`

Received October 23, 2006
Final Accepted June 27, 2007